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*PINHOLE PROBE EXPERIMENTS WITH MASSIVE AIR COLUMNS**

By Carl Barus

Department of Physics, Brown University

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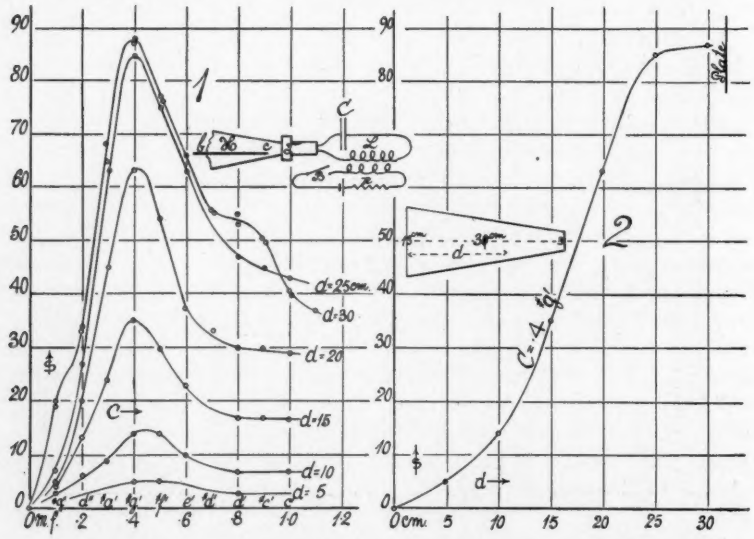
1. *Apparatus. Broad Horn (19°).*—Hitherto in my work short and slender pipes (5 to 10 cm. long, 1 cm. in diameter) were used in connection with the telephones. The present investigation of broad and long pipes contrasts with this and leads to a number of interesting results bearing on the acoustic pressure (s) indicated by the pinhole probe. As first found, the phenomena with the conical horn were so much simpler than the complicated graphs for the cylindrical pipe, that the use of the former in connection with measurements of alternating currents seemed promising. Later work did not bear this out. The horn used (Fig. 1 and 2 inserts) had the conical shape, being 30 cm. deep, 13 cm. in diameter at the mouth and 3 cm. at the base (apex angle 19°). It was attached to a telephone, (see Fig. 1, T and H), as this seemed to be the only way of activating the horn when air currents must be excluded. The attachment was secured through a perforated cork and thin brass tube, 2 cm. long and 1 cm. in diameter, flanged and cemented to the telephone cap. Thus the telephone plate was about 2 cm. behind the flat bottom of the horn. The absence of all constriction might have been desirable, but the data do not show it.

Two methods were used for sounding the horn at all pitches. In the first the secondary of a transformer (1 insert), with known capacity, C, and inductance, L, the capacity (as usual), being variable at pleasure, operated the telephone, T, and the pitches available lay between a and a''. Higher frequencies were tested later. This method is necessarily discontinuous from capacity to capacity. The break, B, in the primary was the common spring platinum contact and it seemed to suffice, care being taken not to change the tension of the spring and modify the frequency factor. The horn sounds with the usual loud blare, characteristic of the pitch of the spring break and the pitch supposedly measured is not heard by the ear.

* Advance note from a Report to the Carnegie Institution of Washington, D. C.

In the second method (Fig. 3, insert), a simple circuit from two storage cells, E, with resistance, R, was periodically broken by the commutator motor device, B, the speed of the motor being controlled by a resistance (electric siren). Contact difficulties at the commutator are here not infrequent. The pitch in this case must be determined by the ear if the method is to be adequately expeditious, but it has the advantage of a continuous method. Frequencies g' to a'' were first available. The lower frequencies, g to a', were added later.

The pinhole probe on a long quill tube (bc, Fig. 1), connecting it with the interferometer U-gauge, was inserted into the horn at the place to be tested, d cm. from the mouth and s, the fringe displacement observed, is proportional to the pressure in the U-gauge.

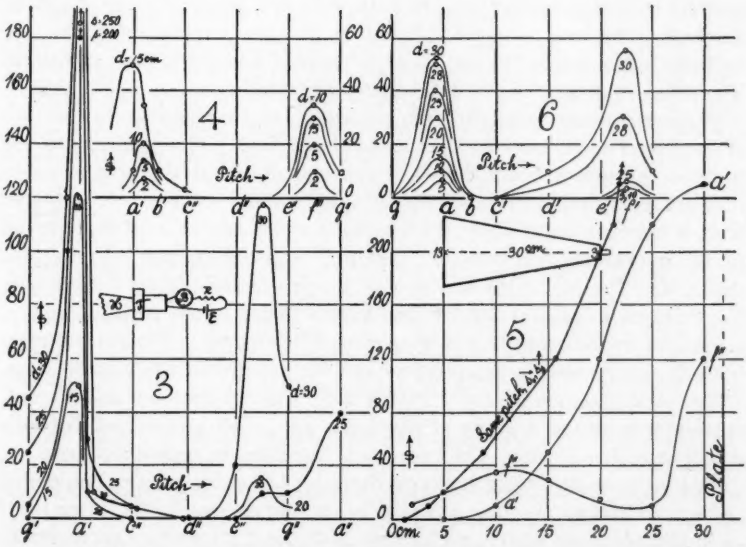


2. *Results for Horn.*—These are given in the graphs, figure 1, as obtained with the transformer method, the abscissas showing the capacities in microfarads with a fixed L (about 38 henry) in the secondary. The ordinates are the acoustic pressures or nodal intensities, s, measured by the pinhole probe when sunk to a depth, d, below the mouth of the horn (s is roughly in 10^{-6} atm.).

What strikes the eye at once is the relative simplicity of this group of curves when compared with similar graphs for other pipes. For all depths, d, below the mouth of the horn, there is a dominant crest at about $C = 0.4$ m.f., though near the mouth ($d = 5.10$ cm.) it seems to shift slightly into large, C-values (flattens). Only in case of $d = 30$ cm. at high pitch

or very low pitch is there a bulge suggesting a crest about $C = 0.1$ m.f. and above $C = 0.8$ m.f., but these vanish even at $d = 25$ cm.

If from these graphs we plot s against d for $C = 0.4$ (near the crest), the curve, figure 2, of very definite character results. It indicates the increase in the nodal intensity (s) of vibration, from the mouth to the base of the horn. The rise is at first accelerated and finally rapidly retarded toward the telephone plate. Throughout much of its extent it is nearly straight. The chief feature of the graphs obtained with electric oscillation is the continuity of sound at all pitches. This continuity, s, in general increasing as pitch falls, will in case of cylindrical pipes occur even without a crest.



If we put $C = 0.4$ m.f. and $L = 0.32 + 0.06$ henry and compute the frequency n as $1/n = 2\pi\sqrt{LC}$ the result is $\#g'$, $n = 408$. If $C = 0.35$ at the crest, $n = 435$ or a' is the pitch. Hence the pitch of the horn, so far as it can be found in this way, lies between $\#g'$ and a'. Meanwhile the pitch of the spring break was about a, its tension being adjusted to secure a maximum of s.

The results obtained with the electric siren, where the pitch is directly and continuously given by the ear, are summarized in figures 3 and 4. Here the currents are more intense and the enormous a' crests for the pipe depths $d = 28$ and 25 cm. run far out of the field. Otherwise their location is in correspondence with its former C-value, and at $d = 10$, 5, 2 cm., it again shifts, but now to higher frequency.

In figures 3 and 4, however, there is a new crest near f'' which does not appear in the preceding graphs. It is, moreover, highly variable with the depth, d, below the mouth of the horn, and is strong only near the telephone plate. The feature of these siren graphs (differing from the electric oscillation graphs) is the occurrence of sharp maxima of sound between long intervals of silence.

In figure 5, the nodal intensity, s, is plotted against the pipe depth, d, for the a' crest and the f'' crest. The a' curve in the present cramped scale is much the same as in figure 2, and is the horn fundamental regarded as a closed pipe. The f'' vibrates as if it were the first overtone with two nodal crests, but enormously dislocated in pitch. With the transformer method it should appear at $C = 0.14$ m.f., and hence if sharp might be overlooked between 0.1 and 0.2 m.f. in this very crowded region. In fact the bulge at $d = 30$ cm., $C = 0.1$ m.f. in figure 1 is thus explained. The other bulge at $C = 0.8$ m.f. is not understood.

In figure 6 I have given the siren graphs for the two and four feet octaves of the broad horn, made at a somewhat later date. This work is difficult because the motor break does not run smoothly but the graphs are a reduced repetition of those for the one and two-feet octaves. As obtained from a one-foot horn, figure 6 probably is a response to overtones evoked by the impressed lower register. Again f'' drops off quickly with the pipe depth, d. Turning to the transformer graphs, figure 1, there is nothing in the f' region to suggest a crest, but a mere decay of the nodal intensity, s. As before the continuity of noise evoked by electric oscillation contrasts with the characteristic sharp crests and flat troughs of the electric siren.

The oscillation graphs for intervals of 0.01 m.f., between 0 and 0.1 m.f. consisted merely of a group of divergent lines with a rapid fall between $d = 30$ and $d = 25$ cm. They need not, therefore, be reproduced here.

The pitch of the spring break in the most favorable position happened to be a; so that an overtone a' would be usually present in the spring itself or in the telephone plate. In fact by changing the tension, and therefore the pitch of the spring, the crests could be shifted from about $C = 0.35$ to even $C = 0.55$. A low C-value for the crest usually accompanying a high intensity (s) value and vice versa. On further change of the spring it was even possible to obtain graphs (for $d = 0.25$), with 3 well-defined crests near $\#g''$, $\#f'$ and between d' and $\#d'$. The graphs thus depend essentially on the pitch of the spring and the simple cases of figure 1 are the result of resonance between the spring break and the several oscillations which it unlooses.

To return to figure 1: Of the three coupled vibrating systems, two, viz., the spring break and the horn, are in the same key (frequency taken as a and a'); the third, the electric oscillation, is varied nominally over a wide range of frequencies (c'' to c'). It seems probable that the massive

air column of the horn, reacting on the telephone plate, forces the electric oscillation to remain at the horn frequency a', with different amplitudes passing a maximum when the electric oscillation is also at a'; i.e., in resonance. Hence the difference of figures 1 and 3, since, in the latter, no reacting mechanism is available. The continuous noise in figure 1 throughout all frequencies must thus be regarded as a case of forced vibration of the L circuit with synchronism at a' while the horn is in turn governed by the spring break. At the same pitch the nodal intensity of the free system (siren, s_s) increases much faster than the nodal intensity of the forced system (electric oscillation, s_0), as shown by the graph in figure 5, where the horizontal scale s_0 is increased four times.

Finally the decreasing nodal intensity (s) toward the mouth of the horn is enhanced by its increasing sectional area. This somewhat obscures the linear relation of the fringe displacement, s, to the corresponding nodal pressure, as I shall indicate in a subsequent paper in which large cylindrical pipes are tested, for s measures the potential energy of the stationary wave at the pinhole point of the probe.

THE FINE STRUCTURE OF THE HELIUM ARC SPECTRUM

By William V. Houston*

California Institute of Technology

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Ever since its discovery the spectrum of helium has been thought to consist of singlets and doublets. Recently, however, spectroscopic theory has indicated that it should consist of singlets and triplets since helium is a two-electron system. Various attempts have been made to resolve the doublets into triplets,[1] but up to the present there have been no conclusive results. Within the last few months interest has been revived in these lines by the theoretical work of Heisenberg,[2] which predicts a triplet structure with unusual separation ratios. It is now possible to show that the helium lines really have a structure similar to that predicted by Heisenberg.

The resolution of these lines has been made possible by the use of the compound Fabry-Perot interferometer[3] for the diffuse series line 5876, and the use of gold films on the simple interferometer for the sharp series line 7065. It is also necessary to cool the discharge tube in liquid air and to use a low current density. The use of the gold films was suggested by Dr. Babcock and produces a large increase in efficiency in the red.

The wave-lengths of the three components of these lines are given in

table 1. They were determined with reference to the singlet lines 6678.149 and 5015.675 as standards, and are probably correct within 0.003 Å. The centers of gravity are very near the values determined by Merrill.[4] The apparent doublet separations of 4713, 4471 and 3888 were also measured and are given in the same table. For these lines the wave-lengths of the weak components were determined by using the measured separations in connection with Merrill's values for the strong components.

TABLE 1

INTENSITY	DESIGNATION	WAVE-LENGTH IN I. A.	$\Delta\nu$
1	$2^3P_0 - 3^3S_1$	7065.707	
			0.992
3	$2^3P_1 - 3^3S_1$	7065.212	
			0.071
5	$2^3P_2 - 3^3S_1$	7065.177	
1	$2^3P_0 - 3^3D_1$	5875.963	
			0.928
3	$2^3P_0 - 3^3D_{12}$	5875.643	
			0.120
5	$2^3P_2 - 3^3D_{123}$	5875.601	
1	$2^3P_0 - 4^3S_1$	4713.373	
			1.038
8	$2^3P_{12} - 4^3S_1$	4713.143*	
1	$2^3P_0 - 4^3D_1$	4471.681	
			1.021
8	$2^3P_{12} - 4^3D_{123}$	4471.477*	
8	$2^3S_0 - 3^3P_{12}$	3888.646*	
			0.286
1	$2^3S_0 - 3^3P_0$	3888.603	

* Merrill's values.

The intensities in the triplets were estimated visually. One of the strong lines was clearly stronger than the other, while these two together are known to have eight times the intensity of the weakest component as shown by D. Burger.[5] Accordingly they were judged to have close to the theoretical values of 5:3:1. The separation of 4713 is very close to the corresponding separation in 7065, while the separation of 4471 shows clearly the effect of the decrease of the $4D$ term over the $3D$. The separation of 3888 is very close to that given by the work of McCurdy.[6]

The fact that two components are so close together in the strong line makes it difficult to distinguish their resolution from a reversal. There are several reasons for believing that we have here a case of true resolution. In the first place the components are not of equal intensity. The phenomenon of unsymmetrical reversal has been recorded frequently, but usually under circumstances which make it probable that more than a single line was present. Furthermore, the discharge tube used as a source was

such that reversal was not likely. It was bent in the form of a U and inserted in a liquid air flask, so that the discharge was always surrounded by liquid air, while the tube through which the discharge was observed extended up out of the liquid air and was thus at a higher temperature. Most conclusive, perhaps, is the fact that the definition improved with a decrease in current and a decrease in temperature as the tube became cooled with liquid air. In fact, without the liquid air, the structure almost disappeared and reversal appeared in 6678. This reversal had disappeared entirely when the plates were taken. The current density used was 0.09 amp. per sq. cm.

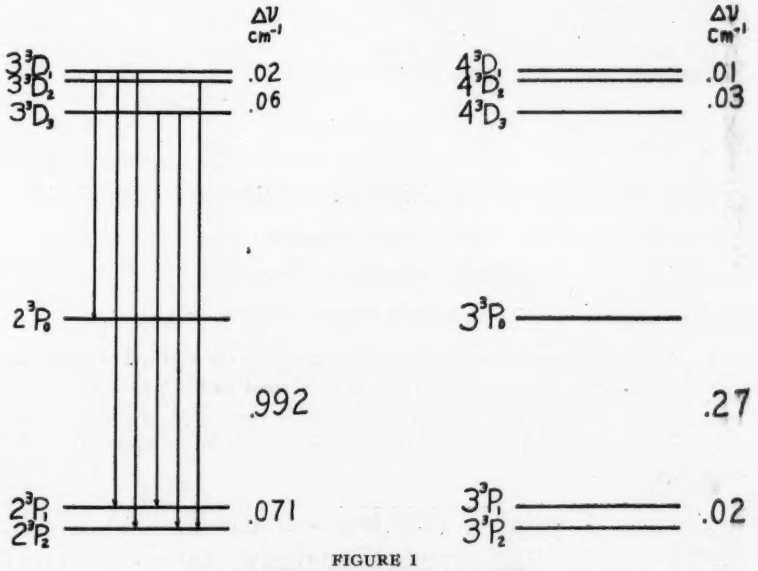

FIGURE 1
Fine structure of energy levels in Helium.

Figure 1 gives the scheme of energy levels which will produce the observed lines. The P levels are entirely inverted, while the D levels are partly so. The theory of Heisenberg requires that both levels be inverted, but the small differences involved may easily be due to the terms neglected in his approximations. The separation ratio of the P term is 14:1 instead of 10:1 as given by the theory. The ratio of the $3D$ separation to the $2P$ separation is of the order of magnitude predicted. The arrangement of the $4D$ terms is assumed to be the same as that of the $3D$, and the separation is roughly determined from the measurement of 4471. The $3P$ term is assumed to be the same in arrangement as the $2P$ term, although its close

components have not been resolved, and its wider separation is determined by the measurements. Thus this work confirms the theory of Heisenberg in its essential points.

In this way the experimental work, by showing that the helium spectrum contains singlets and triplets, confirms the theoretical predictions and thus removes this spectrum from its anomalous position.

* NATIONAL RESEARCH FELLOW.

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[2] W. Heisenberg, *Zeit. Physik*, **39**, 499, 1926.

[3] Houston, *Physic. Rev.*, March, 1927.

[4] Merrill, *Astroph. J.*, **46**, 357, 1917.

[5] D. Burger, *Zeit. Physik*, **38**, 437, 1926.

[6] McCurdy, *Phil. Mag.*, **2**, 529, 1926.

TWO REMARKS ON SCHRÖDINGER'S QUANTUM THEORY

BY PAUL S. EPSTEIN

CALIFORNIA INSTITUTE OF TECHNOLOGY

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1. As our first remark we wish to point out that the method of reduction of Schrödinger's equation which the author used in his treatment of the Stark effect[1] permits also an extremely simple treatment of the relativity effect in the neutral hydrogen atom. The Hamiltonian equation in this case has the form

$$mc^2\sqrt{1 + (\nabla S)^2/m^2c^2} = E' + e^2/r, \tag{1}$$

where e, m are the charge and mass of the electron, c the velocity of light, and where the energy E' includes the inner potential energy mc^2 of the electron ($E' = E + mc^2$). Solving with respect to ∇S:

$$(\nabla S)^2 - [(E' - e^2/r)^2 - m^2c^4]/c^2 = 0. \tag{2}$$

We obtain the wave equation of the problem replacing ∇S by the operator $iK\nabla$ (where $K = h/2\pi$)

$$\nabla^2\psi + [(E' - e^2/r)^2 - m^2c^4]\psi/K^2c^2 = 0. \tag{3}$$

As in the non-relativistic case, we substitute

$$\psi = \chi(r)P^n_{k-1}(\cos\vartheta)\cos n\varphi, \tag{4}$$

and obtain for $\chi(r)$ the equation

$$\frac{d^2\chi}{dr^2}+\frac{2}{r}\frac{d\chi}{dr}+\left[A+\frac{2B}{r}+\frac{p_0^2-k(k-1)}{r^2}\right]\chi=0, \qquad (5)$$

$$A=(E'^2-m^2c^4)/K^2c^2,\ B=e^2E'/K^2c^2,\ p_0^2=e^4/K^2c^2. \qquad (6)$$

This equation can be treated exactly in the same way which we have used in the other case. We put as there

$$\chi=r^{k-1}\exp(\alpha r)M, \qquad \alpha^2=-A. \qquad (7)$$

This leads to the equation for M:

$$\frac{d^2M}{dr^2}+2\left(\alpha+\frac{k}{r}\right)\frac{dM}{dr}+\left[2\frac{\alpha k+B}{r}+\frac{p_0^2}{r^2}\right]M=0. \qquad (8)$$

This equation is again of the hypergeometric type and M can be easily represented as a power series

$$M=\sum a_s r^s. \qquad (9)$$

For the coefficients a_s we obtain the relation

$$[\delta^2+(2k-1)\delta+p_0^2]a_s+2\alpha[\delta+k-1+B/\alpha]a_{s-1}=0. \qquad (10)$$

The coefficient of a_s gives us the exponent of the first term of the expansion (9). The roots of $\delta^2+(2k-1)\delta+p_0^2=0$ are

$$\delta_0=\pm\sqrt{\left(k-\frac{1}{2}\right)^2-p_0^2}-\left(k-\frac{1}{2}\right). \qquad (11)$$

We may use only the upper sign as the function $\chi(r)$ must be finite in the point $r=0$ and as the lower sign would violate this requirement. In the other terms of the sum (9) δ will assume the values $\delta=\delta_0+n$, k and n being integers. The coefficient of a_{s-1} then can be written as $[\delta_0+B/\alpha+n+k-1]$. This coefficient will vanish for a certain value of n, if and only if $\delta_0+B/\alpha+k$ is a negative integral number $-l$:

$$\delta_0+B/\alpha=-(k+l). \qquad (12)$$

On the other hand it is easy to see that, unless the series (9) breaks off, the function $\chi(r)$ will be infinite for $r=\infty$. Relation (12) represents, therefore, the condition of finiteness. Substituting δ_0, B, α from (12), (6), (7) and solving with respect to E, we have

$$E=mc^2\left[1+p_0^2/\left(l+\frac{1}{2}+\sqrt{\left(k+\frac{1}{2}\right)^2-p_0^2}\right)^2\right]^{-1/2}-mc^2 \qquad (13)$$

as found by Schrödinger by a different method.

2. The second remark bears on the problem of the relation of Schrödinger's equation to the theory of light quanta. The fundamental concept introduced by *Louis de Broglie* is that with motion of a mass there is always associated a wave motion. Let the mass, m, be moving without forces in

the direction x, with the velocity $\dot{x}$. Denoting the momentum by $p = m\dot{x}$ we can write the energy equation as

$$p^2/2m = E. \tag{14}$$

The corresponding wave equation results then by Schrödinger's method as

$$\partial^2\psi/\partial x^2 + 8\pi^2 mE\psi/h^2 = 0. \tag{15}$$

Let us now reverse the problem: We consider a plane light wave as given, according to the electromagnetic theory, by the equation

$$\partial^2\psi/\partial x^2 + k^2\psi = 0, \qquad k = 2\pi/\lambda = 2\pi\nu/c. \tag{16}$$

We put the question: Will there be associated with this light wave the motion of a material particle and what will be the momentum and the kinetic energy of this particle?

According to de Broglie the velocities $\dot{x}$ and V (of the wave) are related by the equation $\dot{x}V = c^2$. In the case of light waves we have $V = c$, giving $\dot{x} = c$. Our particles will move with the velocity of light, and this makes a change of equation (15) necessary: For particles moving with the velocity c the energy is expressed as mc^2 and not $m\dot{x}^2/2$. The momentum is $p = mc$ and the energy equation can be written as

$$mc^2 = p^2/m = E. \tag{17}$$

The corresponding wave equation is

$$\partial^2\psi/\partial x^2 + 4\pi^2 mE\psi/h^2 = 0. \tag{18}$$

This must be compared with the form (16). We see that we have to identify

$$4\pi^2 mE/h^2 = k^2 = 4\pi^2\nu^2/c^2. \tag{19}$$

Making use of relation (17) we find

$$E = h\nu. \tag{20}$$

We see that the particles associated with the motion of light waves have the energy $h\nu$ and the momentum $h\nu/c$, i.e., they have exactly the characteristics which Einstein ascribed to the light quanta.

Some interesting considerations on the connection of light quanta with Schrödinger's theory were published by L. de Broglie,[3] but the aspect of the matter presented in the above lines seems to be new. Of course, we regard it as a first and incomplete attempt, as there remain still many difficulties to be elucidated. Especially it should be mentioned that Schrödinger's functions represent a system of stationary waves, while in optics we are particularly interested in progressing waves.

[1] P. S. Epstein, *Physic. Rev.*, **28**, p. 695, 1926.

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[3] L. de Broglie, *Nature*, **118**, p. 441, 1926.

ON KING'S CLASSICAL THEORY OF RADIATION

BY BORIS PODOLSKY

NORMAN BRIDGE LABORATORY OF PHYSICS, PASADENA, CAL.

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1. *Introduction.*—In his paper, "Gyromagnetic Electrons and a Classical Theory of Atomic Structure and Radiation,"[1] Louis V. King endeavors to give a physical image of the quantum mechanism. Ever since quantum phenomena became definitely recognized many attempts were made to picture their mechanism.[2] In the case of King's theory there is an additional appeal to a scientific mind in the fact that his picture seems to unify the quantum phenomena with the classical electro-magnetic theory of Maxwell-Lorentz. Briefly stated the essentials of King's theory are as follows:

An electron is assumed to be a rigid sphere carrying a surface charge uniformly distributed and rigidly attached to the surface.

The electron is assumed spinning with constant intrinsic angular velocity Ω, the same for all electrons. The shape of the electron is assumed unchanged by rotation.

However, when the electron is given a velocity of translation, v, it is assumed to experience a Lorentz-Fitzgerald contraction in the direction of motion. The total electromagnetic energy of the deformed electron is then calculated. Allowing an amount $mc^2\alpha^2/4$ + constant for the work done by the boundary stresses of the electron, King arrives at the following expression for the entire energy of the deformed spinning electron moving with a uniform velocity, v.

$$T = mc^2 + {}^1/_2(Aw_1^2 + Bw_2^2 + Cw_3^2) + \text{constant}, \qquad (1)$$

where

c is the velocity of light;

$\alpha^2 = 1 - \beta^2 = 1 - v^2/c^2$;

w_1, w_2, w_3 are the components of the angular velocity in three mutually perpendicular directions, the first direction being chosen along v;

m is the mass of the moving electron assumed uninfluenced by rotation, i.e.,

$m = m_0/\alpha$, where m_0 is the rest mass;

A, B and C are the three principal moments of inertia, for which King obtains

$$A = ma^2(1-3\beta^2/5)/3(1-\beta^2) \qquad (2)$$

$$B = C = ma^2(1 + 4\beta^2/5)/3 \qquad (3)$$

where a is the radius of the electron before deformation.

On the basis of the above expression for the total energy King assumes that the electron moving with a constant velocity, v, behaves like a rigid body having moments of inertia A, B and C. Now, since the axis of spin in general does not coincide with any of the principal axes, he is led to the conclusion that the electron would precess.

Designating by h a quantity given by

$$h = 5\pi c^2 m_0/2\Omega, \tag{4}$$

using Euler's equations for rotation of rigid bodies, neglecting terms of the order of β^2 and higher, King finally obtains for the frequency of precession, ν, the relation

$$h\nu = \frac{1}{2} m_0 v^2. \tag{5}$$

King further assumes that when radiation impinges upon matter only those electrons are affected for which the precessional frequency, ν, is substantially equal to the frequency of the incident radiation, in which case the electron absorbs energy performing precession with increasing amplitude until it is finally ejected. In the process of ejection part of the energy of the electron P is spent in supplying the energy necessary for its removal, the remainder becoming the kinetic energy of the ejected electron, so that

$$\frac{1}{2} m_0 v^2 = P + \frac{1}{2} m_0 V^2, \tag{6}$$

V being the velocity of the ejected electron. By substituting from (6) into (5) one obtains the photo-electric formula

$$h\nu = P + \frac{1}{2} m_0 V, \tag{7}$$

where h is identified with Plank's constant.

The line spectra are explained in this theory by assuming that only those atoms are radiating in which the precessional angular velocity $2\pi\nu$ is related to the angular velocity in the orbit $n = v/r$ by the relation

$$2\pi\nu = sn. \qquad (s = 1, 2, 3, \ldots .). \tag{8}$$

On reading King's paper one is continually impressed by the ingeniousness of the theory, but, unfortunately, there are grave objections against it.

2. *Objections.*—If the electron is assumed to behave as a rigid body and acquire a precession, it could not absorb energy and maintain the frequency of precession. This can be seen by expressing the frequency in terms of the intrinsic spin.

Since two of the moments of inertia are equal, i.e., $B = C$, we can use

the theory of symmetric tops.[3] The frequency of precession comes out equal to

$$\nu = \frac{1}{2\pi}\sqrt{2E/B - A\Omega^2/(B - A)} \tag{9}$$

where E is the total energy and the Ω intrinsic spin. If the intrinsic spin is assumed constant, as is done by King, the energy cannot change without a change of frequency of precession. Thus, photo-electric phenomena with monochromatic incident radiation would be impossible.

There is, however, another and much more important objection: We can easily show that a spinning electron moving with uniform translational velocity cannot have a precession at all. This is at once evident in Einstein's theory of relativity as the electron is not deformed to an observer moving with it; then, since it would not precess to this observer, it cannot precess at all.

However, King makes use of relativity only to the extent of the Lorentz-Fitzgerald contraction. Therefore, we obtain the same result by the following non-relativistic considerations. In fact, at some instant the angular momentum of the electron, as given by its three components, is

$$M = (Aw_1, Bw_2, Cw_3), \tag{10}$$

the components being taken in the three mutually perpendicular directions fixed in space. This can be done by choosing one of the axes along the axis of symmetry, which permanently coincides with the direction of the velocity, v. At some subsequent instant

$$M' = (A'w_1', B'w_2', C'w_3'). \tag{11}$$

Since it is assumed that there are no external forces

$$M = M'. \tag{12}$$

We must, therefore, have

$$Aw_1 = A'w_1', Bw_2 = B'w_2', \quad Cw_3 = C'w_3'. \tag{13}$$

Now, since our choice of axes is the same as in King's paper, the equations (2) and (3) show that A, B and C will remain unchanged, so that

$$A = A', \ B = B', \ C = C'. \tag{14}$$

When these are substituted into (13) we have

$$w_1 = w_1', \qquad w_2 = w_2', \qquad w_3 = w_3', \tag{15}$$

so that the three components of the angular velocity referred to the axes fixed in space remain unchanged, i.e., *there is no precession*. King derives a precession by using Euler's equations which are applicable to *rigid* bodies with the axes *fixed in the body*. The essential assumption in these

equations is that the *principal axes of the body rotate with the body*, while in the case of a deformed electron moving in a straight line, the direction of the axis of symmetry is always that of the velocity, i.e., *fixed in space*. Thus, the electron does not behave as a rigid deformed sphere.

3. *Conclusion.*—We must, therefore, conclude that King's picture is incorrect. Moreover, it seems hardly desirable. In the modern theory the frequency of precession of the spinning electron (due to causes other than those discussed by King) has found its proper place. It accounts[4] for the normal doublets and triplets, that is, for a small change of the frequency and not for the whole effect.

[1] Mercury Press, Montreal, Canada.

[2] See, for instance, J. J. Thomson, *Phil. Mag.*, April, 1919.

[3] See, for instance, E. T. Whittaker, "Analytical Dynamics," VI, 69.

[4] See, for instance, Uhlenbeck and Goudsmit, *Naturwissenschaften*, **13** (1925), p. 953; *Nature*, **117** (1926), p. 264; *Physica*, **6** (1926), p. 273; Heisenberg and Jordan, *Z. Physik*, **37** (1926), p. 263.

ABSORPTION BANDS IN NITROGEN

BY H. SPONER

PHYSICAL INSTITUTE, GÖTTINGEN

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The excitation potentials of the nitrogen molecule have recently been determined by the electron impact method combined with spectroscopic observation.[1] In my previous measurements I had observed the excitation potential of the O-O band of the second positive group of nitrogen and I had calculated from these measurements and from the term scheme of R. T. Birge the corresponding potentials for the O-O bands of the first and fourth positive groups. Kneser had obtained the excitation potentials of the strongest bands of the first positive group. From these measurements the first excitation potential of the nitrogen molecule was given as 8.0 ± 0.3 volt. The band system corresponding to a transition from the energy level of 8.0 volts to the normal state is unknown and, therefore, a study of the absorption in nitrogen was carried out. The measurements have been made in the Department of Physics of the University of California, Berkeley, and as the time of my stay in Berkeley was limited, the investigation could not be finished and hence this report should be regarded only as preliminary.

The way in which the experiment was carried out is described in a paper being published in the *Zeitschrift für Physik*. I only want to mention that the final experiments were done with the new continuous light source for the extreme ultra-violet described by Lyman[2] and that the spectrograph

used for the last absorption experiment was the apparatus of Dr. Hopfield, which he had kindly offered me, as the experimental conditions in his room were very convenient and as my time was so limited. The excellent grating of his spectrograph was ruled by Prof. Wood.

The first film was obtained with the kind help of Dr. Hopfield. It contained five photographs taken at different pressures. By a mischance which occurred in the laboratory the film was ruined before it could be measured. A second film contained a photograph taken at 100 mm. pressure. Two band systems could be measured, one beginning at 1544 Å and the other beginning at 1450 Å. As the O-O band of the new absorption system was to be expected at about 65,000 cm.$^{-1}$ according to the electron impact measurements, and as the frequency differences of the system beginning at 1544 Å seemed to agree with the experimental limits of error with the frequency differences of the initial state (absorption) of the first positive group of nitrogen, it was concluded in a preliminary note to *Nature*[3] that the system beginning at 1544 Å corresponds to the missing absorption bands in nitrogen. In a kind letter Dr. Birge has informed me that Dr. Hopfield has reproduced the photographs which were on the ruined film and which, were it not for my early departure, would have been completed by me in Berkeley. The dependence of pressure has shown that this system does not belong to nitrogen, since it is absorbed only in the discharge tube, and is the fourth positive group of carbon, that is, the absorption spectrum of CO.[4]

The wave-lengths, frequency differences and quantum designations of the other absorption system, beginning at 1450 Å are given in the following table.

λ	ν	DIFFERENCES IN ν	QUANTUM DESIGNATIONS
1449.9	68970		0–0
		1663	
1415.8	70633		0–1
		1643	
1383.6	72276		0–2
		1613	
1353.4	73889		0–3
		1580	
1325.0	75469		0–4
		1559	
1298.2	77028		0–5
		1528	
1273.0	78556		0–6
		1498	
1249.2	80054		0–7
		1475	
1226.6	81529		0–8

The standards which were used for the calculation of the table are H 1215.68; N 1492.61 and N 1742.81. A comparison of these bands with

the band systems so far known proved that the wave-lengths coincide very nearly with those which belong to corresponding bands in the ultra-violet emission system in N_2, measured and analyzed by Birge and Hopfield.[5] I have believed that this system should be ascribed to the term scheme of the NO molecule, because the frequency differences of its initial state (absorption) agreed with the frequency differences of the final state of the so-called third positive group of nitrogen—absorption band system of NO—within the experimental limits of error. But the absorption experiment in nitrogen settled the question in a respect that the system apparently does not belong to nitric oxide but to nitrogen. We have here, and also in the case of the first-mentioned absorption system, similar difficulties as there were at the decision of the question whether the cyanogen bands belong to nitrogen or to cyanogen.

If the new absorption system is due to nitrogen, then we know also the oscillation frequency of the normal state of the nitrogen molecule, because the system was measured in emission by Birge and Hopfield. The first bands had been found by Lyman.[6] Hence the first oscillation quantum of the normal state is 2345 cm.$^{-1}$. A transition to the excited state is combined with a weakening of the binding,[7] since the first oscillation quantum in this state is only 1680 cm.$^{-1}$. An extrapolation of the convergence limit of the bands of the initial state had already been made[8] when the system was still considered to belong to NO. The limiting energy was 11.75 volts. This agrees within the limits of error of extrapolation with the value of the dissociation heat of nitrogen of 11.4 $\pm$ 0.3 volt which has been derived from an hypothesis on the nature of active nitrogen.[1] We must expect that, provided the new absorption system belongs to N_2, the convergence limit of the bands of the normal state gives the dissociation work of the nitrogen molecule. In a private communication I have heard from Dr. Birge that a greater extension of the Birge-Hopfield system has shown that a linear curve for the initial state is unsatisfactory and that he had to use a parabolic equation. The new solution gives 13.65 volts for the limiting energy, which is 11.4 + 2.25 volts, corresponding to one neutral nitrogen atom and to one 2.4 volt[9] excited nitrogen atom for the products of dissociation.

The fact that the Birge-Hopfield bands are due to nitrogen takes away the only evidence we have of a double electron jump in a molecule. The earlier assumption where the bands had found their place in the term scheme of NO had given 13.96 volts for the excitation potential of the O-O band, which is much more than the ionization potential (at about 9 volts).[10] Mulliken (private communication to Dr. Birge and myself) had accordingly suggested that the excitation of this system represents a double electron transition. The occurrence of the bands in absorption makes such an explanation now unnecessary.

It would be interesting to see whether a band system exists connecting with the Birge-Hopfield bands. It must be a system of which the oscillation quanta of the initial state (absorption) agree with those of the final state of the B.-H. bands. I have not been able so far to find such a system.

Comparing the nitrogen molecule with the CO molecule, the new system would have to correspond to the Angström bands in CO, while the B.-H. bands correspond to the fourth positive group of carbon.[11] The comparison shows also that the Cameron bands in CO correspond to the absorption system in nitrogen which is due to transitions between the normal state and the initial state of the first positive group which I have not been able to get in absorption. It is interesting that Hopfield has now got the Cameron bands in absorption.[12]

I have also tried to get the missing system in nitrogen in emission by photographing the afterglow of nitrogen in the extreme ultra-violet. As it is known, the afterglow contains some bands of the first positive group which are characteristic for the afterglow spectrum. The molecule is after the emission of these bands in an excited state (the first electron jump + several oscillation quanta) from which it should return to the normal state. Unfortunately, long exposure times were necessary with the apparatus I have used (60 hours and more), so that the experiments could only be begun. The expected bands cannot be seen on the photographs which have been taken. It would be important to continue these investigations because the non-occurrence of the bands would suggest that the first excited state of the nitrogen molecule might be metastable. This would give immediately an explanation of the negative result of the absorption experiment. Experiments which were originated in connection with these problems are now in progress in the Physical Institute in Berkeley.

In a summary we can say that the ultra-violet band system found by Birge and Hopfield in nitrogen in emission is a true absorption system of nitrogen. It has no relation to the known bands of nitrogen. The absorption X system which is missing in the known term scheme of N_2 has not been found.

I wish to express my best thanks to the International Education Board which has enabled the present investigation in the Department of Physics of the University of California, Berkeley, by the grant of a fellowship. I take pleasure in acknowledging my indebtedness to the Head of the Department, first Prof. E. P. Lewis and afterward Prof. E. E. Hall, and to Prof. Birge and Prof. Hopfield and also the Shop of the Institute for their interest and help throughout the work.

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[11] K. T. Compton and F. L. Mohler, *Report on Critical Potentials*, p. 124; T. R. Hogness and E. G. Lunn, *Physic. Rev.*, **28**, 849(A), 1926.
[12] For CO: R. T. Birge, *Nature*, **117**, 229, 1926; and *Physic. Rev.*, **28**, 1157, 1926.
[13] T. T. Hopfield, Abstract of the Dec. Meeting of the Am. Phys. Soc., 1926.

ACTION OF RADIATION AND PERTURBATIONS ON ATOMS

BY J. C. SLATER

JEFFERSON PHYSICAL LABORATORY

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Several years ago the writer was struck by the remarkable essential similarity between the action of radiation fields and of ordinary perturbations on atomic systems. Since there was then no direct mathematical method of handling radiation problems, it was impossible to exhibit the analogy in a definite way, and the idea was dismissed as an interesting but unprofitable speculation. Schrödinger's equation, however, makes it possible to show the relation very clearly. We shall adopt the method of solving the perturbed problem in terms of the characteristic functions of the unperturbed one,[1] and shall consider this method in the first section. The second section contains the physical discussion of the problem.

1. We start with Schrödinger's equation[2] $(H-E)\psi = 0$, where in the function $(H\text{-}E)$ of the q's, p's, t and E, we are to replace p_k by $\frac{h}{2\pi i}\frac{\partial}{\partial q_k}$, E by $\frac{h}{2\pi i}\frac{\partial}{\partial t}$. We assume that $H = H_0 + H_1$, and suppose that the problem (H_0-E) can be solved, and that its solution is $\psi_0 = \sum(m)c_m{}^0\psi_m$, $\psi_m = e^{\frac{2\pi i}{h}E_m t}u_m(q_k)$, where the $c_n{}^0$'s are arbitrary constants, and the E_m and u_m are characteristic numbers and characteristic functions, respectively. Now we try to find a solution of the original problem of the form $\psi = \sum(m)c_m\psi_m$, where now the c_m's are no longer constants, and are no longer completely arbitrary. Our equation is $(H_0 + H_1-E)\sum(m)c_m\psi_m = 0$. We assume that $(H_0 + H_1)$ does not contain E explicitly; then since c_m is a function of t alone, $(H_0 + H_1)$ commutes with it. On the other hand, since c_m depends on t, E does not commute with it, but we have $Ec_m - c_mE = \frac{h}{2\pi i}\frac{dc_m}{dt}$. Thus $\sum(m)(H_0 + H_1 - E)c_m\psi_m = \sum(m)$

$\left\{ c_m(H_0 + H_1 - E)\psi_m - \frac{h}{2\pi i}\frac{dc_m}{dt}\psi_m \right\} = 0$. Using $(H_0 - E)\psi_m = 0$, this is $\sum(m)\frac{dc_m}{dt}\psi_m = \sum(m)c_m\frac{2\pi i}{h}H_1\psi_m$. We multiply in front by ψ_n^*, the conjugate of ψ_n. Then $\sum(m)\frac{dc_m}{dt}\psi_n^*\psi_m = \sum(m)c_m\frac{2\pi i}{h}\psi_n^*H_1\psi_m$. Next we integrate over the space coördinates. On this integration we have $\int\psi_n^*\psi_m dv = \delta_{nm}$, $\int\psi_n^* H_1\psi_m dv = e^{\frac{2\pi i}{h}(E_m - E_n)t}\int u_n H_1 u_m dv = e^{\frac{2\pi i}{h}(E_m - E_n)t}(H_1)_{nm}$. Then the equation becomes $\frac{dc_n}{dt} = \sum(m)b_{nm}c_m$, where $b_{nm} = \frac{2\pi i}{h}(H_1)_{nm}e^{\frac{2\pi i}{h}(E_m - E_n)t}$. Thus we have an infinite set of first order differential equations for the c's. It is worth noting that these equations hold whether H_1 is large or small; although generally we apply them only if H_1 is a small perturbative energy. In the latter case it is useful to develop c_n as a power series in the perturbative potential, the formulas for the successive terms being as given by Dirac, Born and the present author. These formulas can be used at once to calculate absorption in an external radiation field.

In many problems it is enough to know the average value of $\frac{dc_n}{dt}$ over a considerable time, which we may call $\frac{\delta c_n}{\delta t}$. This is given by substituting average values of the b_{nm}'s. The resulting differential equations, instead of having an infinite number of terms, have only a finite number. If H_1 is independent of time, the only terms surviving a time average are those for which $E_m - E_n = 0$; for cases where H_1 involves the time, the reader is referred to the paper on absorption. If the system determined by H_0 is not degenerate, $E_m = E_n$ only if $m = n$. Then $\bar{b}_{nm} = \delta_{nm}\frac{2\pi i}{h}(H_1)_{nn}$, so that $\frac{\delta c_n}{\delta t} = \frac{2\pi i}{h}(H_1)_{nn}c_n$, whose solution is immediately $c_n = c_n^0 e^{\frac{2\pi i}{h}(H_1)_{nn}t}$. Thus the answer to the problem, to the approximation to which we may replace $\frac{dc_n}{dt}$ by $\frac{\delta c_n}{\delta t}$, is $\psi = \sum(n)c_n^0 e^{\frac{2\pi i}{h}(E_n + (H_1)_{nn})t}u_n(q_k)$. This gives the well-known result that the term values of a perturbed non-degenerate system differ from those of the unperturbed system by the average value of the perturbative potential.

In a degenerate system, however, $E_m = E_n$ does not demand that $m = n$. Let us rather assume that each energy value of the unperturbed system is not single, but k-fold; we denote them by $E_{m1} \ldots E_{mk}$, where $E_{mi} = E_{mj} = E_m$. Then since each term bears two indices, matrices

must have four, and we have $\frac{dc_{ni}}{dt} = \sum(mj) b_{ni;\,mj}\, c_{mj}$, $b_{ni;\,mj} = \frac{2\pi i}{h} (H_1)_{ni;\,mj}$ $e^{\frac{2\pi i}{h}(E_{mj} - E_{ni})t}$, $\bar{b}_{ni;\,mj} = \delta_{nm} \frac{2\pi i}{h} (H_1)_{ni;\,nj}$. Thus we have $\frac{\delta c_{ni}}{\delta t} = \sum(j) \frac{2\pi i}{h} (H_1)_{ni;\,nj}\, c_{nj}$. We now have no longer a single differential equation to solve, but a family of k simultaneous ones. We assume a solution $c_{ni} = C_n S_i^n e^{\frac{2\pi i}{h} W_n t}$. (We shall generally omit the index n in S_i^n.) After we divide through by $C_n \frac{2\pi i}{h} e^{\frac{2\pi i}{h} W_n t}$, our equations become $W_n S_i - \sum(j)$ $(H_1)_{ni;\,nj} S_j = 0$, $i = 1, \ldots. k$. These are k homogeneous linear equations in the k variables S_i, which can be solved only if the determinant of their coefficients vanishes. This forms a secular equation of the kth degree for W_n, which in general has k separate roots, which we label W_{nj}, $j = 1, \ldots. k$. Corresponding to each of these, we find from the equations a different set of S's; say S_{ij}, $i = 1, \ldots. k$, corresponding to the jth value W_{nj}. Then a particular solution of our problem is $c_{ni} = C_{nj} S_{ij} e^{\frac{2\pi i}{h} W_{nj} t}$, where C_{nj} is arbitrary, and the complete solution is $c_{ni} = \sum(j) C_{nj} S_{ij} e^{\frac{2\pi i}{h} W_{nj} t}$. Thus we have $\psi = \sum(ni) c_{ni} e^{\frac{2\pi i}{h} E_n t} u_{ni} = \sum(nij) C_{nj} S_{ij}^n e^{\frac{2\pi i}{h}(E_n + W_{nj})t} u_{ni} =$ $\sum(nj) C_{nj} e^{\frac{2\pi i}{h}(E_n + W_{nj})t} \sum(i) S_{ij}^n u_{ni}$.

The mathematical steps in the procedure just described are closely related to those found in Heisenberg's and Schrödinger's methods[3] of treating degenerate systems, in which the perturbed problem is not solved in terms of characteristic functions of the unperturbed system, but in terms of the linear combinations of them $u_{nj}' = \sum(i) S_{ij}^n u_{ni}$. They prove that S_{ij} forms an orthogonal matrix, so that this is an orthogonal linear combination. Our final solution for ψ exhibits directly the way in which these functions enter the final result; it can be written $\psi = \sum(nj) C_{nj}$ $e^{\frac{2\pi i}{h}(E_n + W_{nj})t} u_{nj}'$. Thus the constants C_{nj} which we have found are simply the constants corresponding to the state nj in Heisenberg and Schrödinger's form of solution.

2. The solutions which we have found for the quantities c_n in the perturbed system may now be used to compute $c_n c_n^*$, representing the fraction of the systems in the corresponding quantum states.[4] In the first place, we recall the problem of radiation and absorption. There $c_n c_n^*$ changes with time, and its time rate of change can be interpreted as coming from systems which enter or leave the state on account of absorption or emission of radiation. We compare with it first the problem of the non-degenerate system, where $c_n = c_n^0 e^{\frac{2\pi i}{h}(H_1)_{nn} t}$. This gives $c_n c_n^* =$

$c_n^0 c_n^{0*}$, so that in this case the perturbation results in no change of stationary state. The c's change with time in phase, but not in amplitude.

The degenerate case, however, is different. We have $c_{ni} = \sum(j) C_{nj} S_{ij}^n e^{\frac{2\pi i}{h} W_{nj} t}$, $c_{ni} c_{ni}^* = \sum(jl) C_{nj} C_{nl}^* S_{ij}^n S_{il}^n e^{\frac{2\pi i}{h}(W_{nj} - W_{nl})t}$. Here $c_{ni} c_{ni}^*$ is no longer constant (although $\sum(i) c_{ni} c_{ni}^*$ is, as we immediately prove from the orthogonal character of the S's) but has a multiply periodic oscillation, with the various difference frequencies between the perturbed energy levels into which the original unperturbed level E_n splits up. The frequency of this oscillation or pulsation is thus proportional to the strength of the perturbing field. This case seems very closely analogous to that of radiation, where also the $c_n c_n^*$'s change at a rate depending on the strength of the external field (the fact that it depends in that case on the square of the field is hardly of essential significance). In both cases of course in the limiting problem where the external field is absent, all the $c_n c_n^*$'s remain constant. But in the problem of radiation, energy flows into or out of the system, while in the degenerate system it flows from one degree of freedom to another. Neglecting for the moment the perturbative energy, except as it affects the time variation of the $c_n c_n^*$'s, we have for the energy connected with atoms in the (ni)th state, $c_{ni} c_{ni}^* E_n$, which, therefore, undergoes the same pulsations as the quantity $c_{ni} c_{ni}^*$ itself; on the other hand, the total energy connected with all the states ni, $i = l, \ldots . k$, is $\sum(i) c_{ni} c_{ni}^* E_n = \sum(j) C_{nj} C_{nj}^* E_n$, which is constant, showing that one state loses energy as another gains it.

These remarks about pulsation of energy become more vivid if we fix our attention on a particular degenerate problem, that of two identical systems coupled together, discussed by Heisenberg.[5] We denote the first system by A, the second by B. The unperturbed problem is degenerate, because interchange of the quantum numbers of the systems A and B does not change the energy. Each energy level is then two-fold, so that $k = 2$. If we label the states by two subscripts, the first representing the system A, the second B, we have then $E_{nm} = E_{mn}$ (meaning that the state where A has quantum number n, B has m, has the same energy as the one where A has m, B has n). In the perturbed problem, we have $c_{nm} c_{nm}^* = \sum(j,l = 1,2) C_{nm,j} C_{nm,l}^* S_{nm,j} S_{nm,l} e^{\frac{2\pi i}{h}(W_{nm,j} - W_{nm,l})t}$ with a similar equation for $c_{mn} c_{mn}^*$ where, as Heisenberg has shown, $W_{nm,1} = (H_1)_{nm,nm} + (H_1)_{nm,mn}$, $W_{nm,2} = (H_1)_{nm,nm} - (H_1)_{nm,mn}$, and $S_{nm,1} = S_{nm,2} = S_{mn,1} = -S_{mn,2} = \frac{1}{\sqrt{2}}$. Thus

$$c_{nm} c_{nm}^* = \frac{1}{2}\{C_{nm,1} C_{nm,1}^* + C_{nm,2} C_{nm,2}^*\} + \frac{1}{2}\{C_{nm,1} C_{nm,2}^* e^{\frac{2\pi i}{h}(2(H_1)_{nm,mn})t} + C_{nm,2} C_{nm,1}^* e^{-\frac{2\pi i}{h}(2(H_1)_{nm,mn})t}\}$$

and $c_{mn}c_{mn}^*$ has the same value except that the second bracket has the opposite sign. Thus, pulsations take place between, for example, the state where the quantum number of system A is n, that of system B is m, and the state where that of A is m, of B n. The time rate of change of $c_{nm}c_{nm}^*$ represents the rate of increase of the fraction of cases in which A is in the nth state, B in the mth. That is, adopting the notation of change of state, in unit time there will be a certain fraction $\left(\frac{d}{dt}\, c_{nm}c_{nm}^*\right)$ of the systems in which there will be a double jump, the system A changing its quantum number from m to n, and B from n to m, so that the systems (nm) will be increased by the amount $\left(\frac{d}{dt}\, c_{nm}c_{nm}^*\right)$, and those (mn) will be decreased by the same amount. The fraction of systems A in a given state, without reference to the state of the second, is $\sum(m)c_{nm}c_{nm}^*$. Its time derivative, $\sum(m)\,\frac{d}{dt}\, c_{nm}c_{nm}^*$, represents the time rate of increase of the fraction of systems A in the nth state, the mth term representing the transitions from the mth to the nth state. Each transition of the system A in which we are interested is coupled with an equal and opposite transition of the system B. But the second system need not concern us; we simply have the system A, on which we fix our attention, performing transitions under the influence of B, the rate of transition depending on the strength of coupling.

Next consider the problem of many coupled systems. The coupling energy H_1 will in general consist of many terms, each containing the coordinates of two systems only. Corresponding to this, there will be transitions in which pairs of systems interchange their quantum numbers. Any particular system, as before, will have probabilities of transition to all other states, but now, while there must be an accompanying opposite transition of some other system, there is large latitude as to which that other system will be. That is the only essential difference as far as the transition probabilities are concerned.

Finally, let us go to the case of an atom coupled to a radiation field. It has been shown by various writers[6] that a field may be profitably treated as a vibrating system, having all possible characteristic numbers or frequencies. Thus, in particular, it matches the energy levels of the atom, so that the problem of the atom and field without coupling is a degenerate one. When the coupling is put in there are double jumps, in which the atom changes its stationary state, and some degree of freedom in the field changes its state by an equal and opposite energy to keep conservation. The situation seems to differ in no essential detail from the ones we have been considering. And yet in the case of emission and absorption of radiation, we have strong reason for thinking (photoelectrons, Bothe and Geiger's experiments on Compton effect) that there actually are physical, discon-

tinuous transitions. It seems difficult to escape the conclusion that inside loosely coupled degenerate systems there must be internal transitions, in which electrons interchange their quantum numbers, of just the same degree of objective reality as ordinary external transitions, whatever that may be. Of course, our feeling for generality forbids us to stop here. There is certainly no distinction between loosely coupled degenerate systems, and systems with strong coupling, such as the different electrons in a tightly made atom. All energy terms can be regarded as large perturbations of the degenerate systems formed by removing all forces. This suggests the rather wild speculation that even the large frequencies inside the atom are connected with transitions from one internal state to another; which is very nearly a *reductio ad absurdum* of our idea. The escape seems to be that, if transitions have a real physical meaning, there must be some definite way of computing how often they occur. The method we have used would give a different result with every different choice of characteristic functions. If we used Heisenberg's characteristic functions with the coupled systems, we should find no transitions at all. Probably we could use an analogous method with an atom coupled to a radiation field and find no transitions there either. Yet we know that transitions occur in that case. Apparently there is some criterion, as to the proper set of characteristic functions to use, that is still missing. Whether a correct assignment of these functions would lead to transitions in the case of ordinary radiation problems, and not in coupled systems, can hardly be decided offhand.

It is interesting to consider the spectra of the series of problems which we have discussed. The frequencies which appear in the spectrum are by hypothesis those present in $\psi\psi^*$. These are differences of frequencies present in ψ. But it makes no difference whether the frequencies occurring in ψ are present on account of oscillation of ψ_n or of c_n. Thus in the non-degenerate system, where $c_n = c_n^0 e^{\frac{2\pi i}{h}(H_1)_{nn}t}$, and $\psi = \sum(n)c_n e^{\frac{2\pi i}{h}E_n t} u_n = \sum(n)c_n^0 e^{\frac{2\pi i}{h}(E_n + (H_1)_{nn})t} u$, the frequencies of the spectrum are differences of terms $(E_n + (H_1)_{nn})/h$. The spectroscopic terms, whose differences give spectral frequencies, are simply the frequencies actually occurring in ψ; we may perform a resolution of ψ in Fourier integral, and all the frequencies occurring in this resolution are terms in the spectrum. The application of this principle to the case of an atom in a radiation field is particularly interesting. There, on account of the irregularly changing amplitudes of the c's, the Fourier spectrum of ψ will show each term which, in the unperturbed state, was a sharp frequency, to be broadened. The amount of broadening will be directly proportional to the time rate of change of the c's, or the probability of leaving the stationary state. Thus, in the spectrum, each line will have a breadth which is the sum of contri-

butions from the two terms, the contributions being proportional to the probability of leaving the states in question. This is just the result to which the writer was led earlier from somewhat different physical hypotheses.[7] In considering the broadening of lines, it is also interesting to treat the radiation problem as the limiting case of the interaction of an infinite number of systems all having the same radiation frequency. With two identical interacting systems, each energy level is split into two; as the number of systems increases, the number of component levels also increases, without limit. Thus in the radiation problem, there are an infinite number of levels clustered together, giving the effect of a broadened line. In all cases the amount of displacement of the levels is proportional to the time rate of change of the c's. Our two ways of considering broadening thus coincide in their results.

A question has probably occurred to the reader, in connection with the hypothesis which we mentioned earlier, that the differences between the new terms into which a degenerate term is split by the action of a perturbation measured rates of transition of parts of a system from one state to another. Such an hypothesis often will demand transitions occurring so rapidly that an atom stays in a state only during the time of a few vibrations of its principal frequency before making a transition to another state. The question is this: Would not such frequent transitions produce an enormous broadening of the lines? My answer to this would be the following: We must make a complete separation between the idea of atoms with stationary states and transitions on the one hand, and continuous wave phenomena on the other, assuming only a statistical connection between them as Born does. On the one hand there is the discontinuous world of atoms in stationary states, of radiation in light quanta; on the other the continuous world of Schrödinger's waves ("virtual oscillators"), in place of atoms, and some sort of continuous "virtual" field to represent radiation. Eventually we may be able to devise laws for the discontinuous world which, by the application of statistical methods, will give the continuous laws as averages; but for the present we do not know how and many problems can be answered only by the continuous method. The problem of interference is one of the most striking of these questions; and the broadening of spectral lines, being really an interference in time rather than in space, belongs in this category. Thus we must trust our continuous theory completely in this point, just as we do in ordinary interference. It says that these internal transitions of pulsation (if they occur), produce, it is true, an effect on the spectrum proportional to the rate of transition; but the effect is a sharp shift of the term, rather than broadening. The simplicity of the phenomenon arises from the simplicity of the problem. Radiation produces broadening rather than a simple shift on account of the complication of that effect, and the many degrees of freedom involved.

Thus it seems impossible to draw any conclusions from the nature of the spectrum against our hypothesis; and, of course, it is equally impossible to draw any conclusions favoring it. It is a question which could be settled conceivably only by such direct experiments on single atoms as those of C. T. R. Wilson, of Bothe and Geiger or Compton. As far as ordinary statistical phenomena are concerned, it does not, and in the nature of things cannot, make any difference.

[1] Dirac, *Proc. Roy. Soc.*, **112**, p. 674, 1926; Born, *Zeitsch. Physik*, **40**, p. 172, 1926; Slater, these PROCEEDINGS, January, 1927.

[2] Schrödinger, *Ann. Physik*, **79**, p. 734; **81**, p. 109, 1926.

[3] Born, Heisenberg and Jordan, *Zeitsch. Physik*, **35**, p. 585, 1926; Schrödinger, *Ann. Physik*, **80**, p. 448ff., 1926.

[4] Born, *Zeitschr. Physik*, **38**, p. 803, 1926.

[5] Heisenberg, *Ibid.*, **38**, p. 411, 1926.

[6] Born, Heisenberg and Jordan, loc. cit., p. 606. Other references are given in that place.

[7] Slater, *Physic. Rev.*, **25**, p. 395, 1925.

THE PERIODICITY OF PHOTOELECTRIC THRESHOLDS

BY GEORGE B. WELCH

DEPARTMENT OF PHYSICS, CORNELL UNIVERSITY

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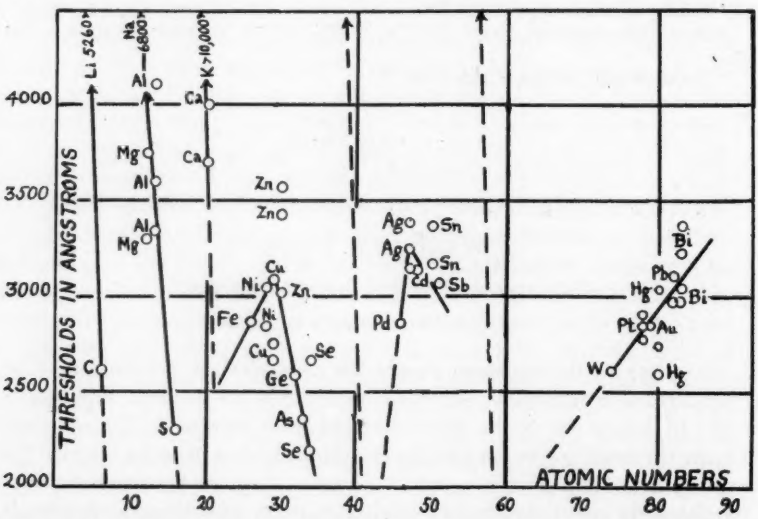
According to the quantum theory the photoelectric threshold may be considered as a measure of the energy taken from the incident radiation in order to detach one of the outer electrons from its atom. The elements having the most loosely bound electrons should permit detachment with the expenditure of minimum amounts of energy. It should be expected, therefore, that the curve showing the relation between photoelectric thresholds and atomic numbers should exhibit a certain periodicity with maximum values for the alkali metals, and that values should show a decrease as one proceeds towards the electronegative elements. Using the present available data, in the accompanying graph, this is found empirically to be the case.

It will be noticed that the curve shows the same general characteristic, a decrease in threshold values as the elements become more electronegative, for the first two, or "short," periods in Mendeleef's table of the elements. Although, in the first period, only two values are plotted, an investigation of the threshold for Be, now in progress, indicates that it will assume a position on the curve.

In the cases of all the alkali metals the curve shows strong maxima.

For three of these, Li, Na, and K, experimental data on the thresholds are available. In the cases of Rb and Cs there is available only the work upon sensitivity and the selective photoelectric effect conducted by observers such as Elster and Geitel,[1] Lindemann,[2] Pohl and Pringsheim[3] and Cornelius.[4] These have found effects with radiation of sufficiently long wave-length to indicate that the thresholds must lie in the infra-red region. Furthermore, Pohl and Pringsheim[3] have shown a maximum effect for Ba corresponding approximately to that of Li. This evidence is some justification for indicating—with broken lines—the probable course of the curve at the beginnings of the fourth and fifth periods.

It will be noticed that in the first and second "long" periods the curve shows a new characteristic: there are minor peaks for Cu and Ag which



might have been expected from the arrangement of the valence electrons in these atoms. In the third "long" period, however, there is no peak for Au; the elements following it seem to possess increasing threshold values. For this apparent deviation from periodicity no explanation can, at present, be offered.

In the "long" periods there is no experimental evidence for locating the minimum points on the curve, and the parts so indicated with broken lines, must be considered as wholly tentative.

On the whole, the experimental conditions have varied widely with the different observers from whom the data used in the graph were obtained, and, under the circumstances, the amount of agreement must be considered as surprisingly good. There remains much work to be done in measuring

the thresholds, for new elements and for those previously determined, under essentially identical experimental conditions.

The references from which the plotted observations are taken are given in the footnotes below.[5-17] For convenience a list of the elements used by the different observers is also included.

[1] J. Elster and H. Geitel, *Ann. Physik*, **43**, 225 (1891); **52**, 433 (1894).
[2] F. A. Lindemann, *Verh. D. Phys. Ges.*, **13**, 482 (1911).
[3] R. Pohl and P. Pringsheim, *Ibid.*, **11**, 1039 (1910).
[4] D. W. Cornelius, *Physic. Rev.*, **1**, 16 (1913).
[5] R. Pohl and P. Pringsheim (Bi, Cd, Hg, K, Pb), *Verh. D. Phys. Ges.*, **13**, 474 (1911).
[6] O. W. Richardson and K. T. Compton (Al, Bi, Cu, Mg, Na, Pt, Zn), *Phil. Mag.*, **24**, 575 (1912).
[7] A. Ll. Hughes (As, Bi, C, Ca, Cd, Mg, Pb, Sb, Se, Zn), *Phil. Trans. Roy. Soc.*, **A212**, 265 (1912).
[8] S. Werner (Ag, Au, Bi, Cu, Ni, Pd, Pt), "Dissertation," Upsala (1914).
[9] R. A. Millikan (Li, Na), *Physic. Rev.*, **7**, 355 (1916).
[10] O. Stuhlmann (Pt), *Ibid.*, **15**, 549 (1920).
[11] M. J. Kelly (S), *Ibid.*, **16**, 260 (1920).
[12] R. Suhrmann (Ag, Al), *Ann. Physik*, **67**, 43 (1922).
[13] F. G. Tucker (Pt), *Physic. Rev.*, **22**, 574 (1923).
[14] R. Hamer (Ag, Al, Bi, C, Ca, Cd, Cu, Fe, Ni, Pb, Pt, Se, W, Zn), *J. Opt. Soc. Amer.*, **9**, 251 (1924).
[15] S. Taubes (Hg), *Ann. Physik*, **76**, 629 (1925).
[16] C. B. Kazda (Hg), *Physic. Rev.*, **26**, 643 (1925).
[17] G. B. Welch (Ge), *J. Opt. Soc. Amer.*, 1927 (in press).

MICROSPOROGENESIS IN THE CUCUMBER

BY L. F. HEIMLICH

VALPARAISO UNIVERSITY

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Many varieties of the cucumber have been in cultivation for a long time. They have been made the subject of many investigations, particularly in respect to methods of cultivation and the control of diseases. Very little information of an anatomical or cytological nature concerning this plant is to be found in literature.

The writer has recently made a study of microsporogenesis in one of the white-spined varieties of *Cucumis sativus* L. The main points of interest drawn from the observations are briefly set forth below.

The primary sporogenous cells function directly as the microspore mother cells. When all the wall layers of the microsporangium are formed, the resting microspore mother cells are about 12 microns in diameter. The nucleus of such a cell measures about 5.8 microns in diameter and the

nucleolus is approximately 2.9 microns in diameter. The chromatin and linin, very meager in amount, are arranged in a fine network of anastomosing threads. Near the end of the resting period the cells increase in size until they measure about 30 microns in diameter. At this time the nucleus measures about 15 microns in diameter and the nucleolus about 4.5 microns in diameter.

A perinuclear zone appears during the early prophases. This zone is at first distinguished from the surrounding cytoplasm by a denser and more deeply staining appearance of radiating fibers. It later becomes chiefly characterized by the presence of numerous mitochondria. Mitochondria and extranuclear nucleoli are at their maximum development in the tetranucleate stage. The very slender chromatin threads of the early prophases tend to pair according to the scheme of parasynapsis. A thick split spireme with paired chromomeres is often found in emergence from synizesis.

The multipolar spindle seems to arise within the nucleus, probably in a manner similar to that described by Allen[1] for Larix, except that no evidence was found showing that the perinuclear zone contributes any spindle fibers in the cucumber.

Many chromosome counts were made in both lateral and polar views of various heterotypic and homœotypic stages. The haploid number of chromosomes is seven. The diploid number, as ascertained in anther primordia and root tips, is 14. The haploid number is the lowest and the only odd number thus far reported for any of the Cucurbitaceae.

Ordinarily in cucumber, the nucleolus disappears by a gradual diminution in size, having very early taken a position just within the perinuclear zone. No nucleoli were observed later than the heterotypic anaphases. In most cases they disappear at an earlier stage.

Budding of the nucleolus is not infrequent, and rarely fragmentation is seen. In some instances the budded part becomes much elongated or vermiform and the globular part approaches a diameter scarcely greater than the thickness of the budded part. Nucleoli disappearing in this very peculiar manner were always observed more or less closely associated with the chromosomes or spindle fibers at multipolar-spindle stages and equatorial-plate stages. In general, their appearance is much like that of some chromosomes. Since they also agree with the chromosomes in their staining reactions, it is quite possible that they may easily be mistaken for chromosomes.

A careful study of this peculiar nucleolar disappearance in cucumber resulted in the demonstration of a graded series of stages of the chromosome-like bodies leading back to early stages of budding nucleoli. Hasty or limited investigation, especially in case such peculiar behavior of the nucleolus is frequent, is likely to result in erroneous conclusions. May

not similar phenomena, at least in part, account for some of the disagreement in chromosome numbers?

A large barrel-shaped spindle is seen between the daughter nuclei during interkinesis. The tetrad nuclei become connected with one another by six spindles. No cell plates were observed during the reduction divisions.

Castetter[2] has recently described cell-plate-formation following both the heterotypic and the homœotypic divisions in the squash. These, he says, are very ephemeral in each case. He also concludes, on the basis of staining reactions, that no spindles consisting of true spindle fibers, are formed after the homœotypic division. The writer's observations on cucumber do not agree with Castetter's on this point. Using the triple stain, the regions between the tetrad nuclei in the cucumber stain violet and have the appearance of spindles made up of true fibers.

Contrary to the opinion held by Strasburger[3] that quadripartition of the pollen mother cell in Cucumis is by cell-plate-formation, all evidence clearly indicates that cytokinesis takes place by furrowing of the plasma membrane, resulting in quadripartition with a tetrahedral or rarely monoplanal arrangement of the microspores.

A small percentage of abortion and abnormal pollen development was noticed.

[1] Allen, C. E., *Annals. Bot.*, **17**, 1903 (281–312).
[2] Castetter, E. F., *Amer. J. Bot.*, **13**, 1926 (1–10).
[3] Strasburger, E., *La Formation Et La Division Des Cellules*, Jena, 1876.

FURTHER STUDIES ON BIOCHEMICAL DIFFERENCES BETWEEN SEXES IN PLANTS[1]

BY SOPHIA SATINA AND A. F. BLAKESLEE

CARNEGIE INSTITUTION OF WASHINGTON, DEPARTMENT OF GENETICS, COLD SPRING HARBOR, N. Y.

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Various tests applied to the green plants and to Mucors in regard to the biochemical differences in sexes have shown that on the average the female plants are superior in a number of reactions tested.[2,3] Among these tests, the reduction reaction seemed to be one of the most promising. A high average difference between the sexes in Mucors was shown when a number of (+) and (−) races were tested in the living condition with Te salts, the (+) races being chiefly the stronger reducers.[3] A few preliminary tests applied to plant extracts showed a greater capacity of female plants to reduce also solutions of $KMnO_4$.[2]

The power of the extracts to reduce $KMnO_4$ has now been tested in a

sufficiently large number of higher plants and Mucors to demonstrate clearly a high average difference between the sexes, female plants being in the majority of cases the stronger reducers. The following method was employed. Two test tubes, each with 5 cc. of a $n/100$ solution of $KMnO_4$ are prepared. This solution is poured at once into 2 other test tubes, each with the same measured quantity of alcoholic extract from male and female plants. The amount of the extract necessary for a clear reaction was determined for each species and ranged from 0.1 to 1 cc. The test tube with the stronger reducing extract changes its color more rapidly. After a given time (3–5 minutes according to the species tested), the color in the test tubes is equalized by adding the necessary quantity of $KMnO_4$ to the tube in which the reduction has proceeded furthest. The number of cc. of $KMnO_4$ added to this test tube shows the relative difference in strength of the reduction brought about by the two extracts.

TABLE 1

RHAMNUS. REDUCTION OF $KMnO_4$ AND MANOILOV REACTION

PLANT NUMBERS		EXCESS $KMnO_4$ REDUCED			MANOILOV REACTION		
♀	♂	♀	♂	REACT. EQUAL	♀	♂	REACT. EQUAL
2	1	0.6	0	—	C+	D+	—
4	3	8.0	0	—	B	D—	—
6	5	5.4	0	—	C+	D−	—
8	9	1.5	0	—	C+	D+	—
10	7	1.5	0	—	C+	C	—
12	14	0	0.6	—	C+	B	—
16	17	4.2	0	—	C+	D−	—
18	19	3.0	0	—	C	C	C
20	15	6.0	0	—	B+	D−	—
22	21	4.5	0	—	B−	D+	—
24	23	0	1.5	—	C+	C	—
26	25	5.4	0	—	C+	D	—
28	33	2.5	0	—	C+	D	—
30	31	2.4	0	—	C+	D−	—
34	35	4.5	0	—	A	C	—
36	32	3.3	0	—	C+	C−	—
38	13	3.0	0	—	B	D	—
29	27	0	0	1	C	C	C
18	18	15	2	1	15	1	2
AVERAGES		3.1	0.1		C+	D+	

Table 1 gives the results obtained from 36 individuals of Rhamnus cathartica tested in pairs. The amount of $KMnO_4$ that must be added to the tube which showed the stronger reduction, in each case, is different. It varies from 0.6 to 8 cc. Out of the 18 pairs, 15 showed a stronger reducing capacity in female individuals and only in 2 pairs were the results reversed. The difference can be seen from the fact that, in equalizing the

colors in the paired tubes, a total of 55.8 cc. had to be added to the female tubes but only 2.1 cc. to the male tubes. This gives an average of 3.1 for the females and 0.1 for the males.

TABLE 2

GREEN PLANTS. REDUCTION OF $KMnO_4$ AND MANOILOV REACTION

SPECIES	PLANTS	REDUCTION $KMnO_4$ STRONGER ♀	♂	(?)	MANOILOV REACTION STRONGER ♀	♂	(?)
Rhamnus	36	15	2	1	15	1	2
Rumex	8	4	—	—	4	—	—
Celastrus	2	1	—	—	1	—	—
Myrica	8	3	1	—	3	1	—
Sassafras	10	5	—	—	5	—	—
Morus	20	8	2	—	9	1	—
Smilax	4	2	—	—	2	—	—
Salix	2	1	—	—	1	—	—
Acer	2	1	—	—	—	—	1
Populus	8	3	1	—	3	1	—
Fraxinus	6	2	1	—	2	1	—
Melandrium	2	1	—	—	1	—	—
Spinacia	12	5	1	—	5	1	—
TOTAL	120	51	8	1	51	6	3
PERCENTAGES		85.0	13.3	1.7	85.0	10.0	5.0

TABLE 3

MUCORS. REDUCTION OF $KMnO_4$ AND MANOILOV REACTION

SPECIES	RACES	REDUCTION OF $KMnO_4$ ♀ STRONGER	♂ STRONGER	MANOILOV REACTION ♀ STRONGER	♂ STRONGER
Rhizopus	10	4	1	4	1
Cunninghamella	8	4	—	3	1
Phycomyces	2	—	1	—	1
Choanephora	2	1	—	1	—
Blakesleea	2	1	—	1	—
Circinella	2	—	1	1	—
Parasitella	2	1	—	1	—
Absidia glauca	2	1	—	1	—
Mucor IV	10	5	—	5	—
Mucor V	2	1	—	1	—
Mucor VII	2	—	1	—	1
Mucor D	2	1	—	1	—
Mucor N	2	1	—	1	—
M. griseocyanus	2	1	—	1	—
M. mucedo	2	1	—	1	—
TOTAL	52	22	4	22	4
PERCENTAGES	—	84.6	15.4	84.6	15.4

Tables 2 and 3 show the results obtained from testing 120 green plants and 52 Mucor races. The percentage of stronger female reducers in both tables were about 85 per cent.

The technique employed in preparing the extracts for these tests was similar to that used in the Manoilov reaction, i.e., the leaves or mycelia

were covered with 60 per cent alcohol and allowed to stand about 2 hours.

In another series, the technique was modified by grinding the material with sand in a mortar, extracting with 60 per cent alcohol and clearing by filtering or centrifuging in order to discover if this change in procedure would bring about reversals in some of the reactions. With this latter technique, 32 individuals were tested: 6 of Rhamnus, 4 of Sassafras, 2 of Mucor N, 2 of Mucor VII, 2 of Phycomyces, 8 of Rhizopus and 8 of Cunninghamella. The absolute strengths of reaction increased markedly, as was to have been expected, but the female extracts remained the stronger reducers. In some cases the difference increased, in others it decreased, but only in one case was a reversed reaction observed.

Thus the results obtained with alcoholic extracts in respect to the reduction reaction correspond to the preliminary tests reported in our earlier paper,[2] in which the material was ground with sand and extracted with distilled water.

Some attempts have been made to determine the types of substances that might be responsible for the differences in reduction capacity shown by male and female plants. Extracts prepared with 95 per cent alcohol reduced the $KMnO_4$ as well as other extracts made with more dilute alcohol and with water. Changes in the strength of alcohol does not spoil the reaction. Preliminary tests have shown also a remarkable thermostability of the reducing substances. Extracts of Rhamnus, Morus and Phycomyces were boiled, and when cold, retested for reduction. In no case was the reaction weakened. These facts suggest that the reducing substances are probably not highly complex compounds such as enzymes which might be changed and rendered inactive by heat and strong alcohol, but that they may belong, in part at least, to some other more stable substances which are widely distributed among plants. Quantitative determinations of carbohydrates and fats, based on calculations of fresh and dried weight of the material used, are now being made. It is hoped they will give a clue to the cause of the differential behavior in respect to the reducing power of male and female plants.

A preliminary investigation on the carotin content of leaves, suggested by the greater average amount of yellow color in female foliage noted in our earlier paper,[2] may be of interest in this connection, since carotin, which has a tendency to unite with oxygen,[4] may be responsible in part at least for our results with green plants. In addition to the 68 plants mentioned in our previous paper,[2] 90 individuals were tested in pairs in regard to the color of their extracts in 60 per cent alcohol. About 66 per cent of the females gave a yellower extract than the males. Quantitative determinations of carotin in leaves of 2 individuals of Morus and 2 of Rumex acetosella were made by Schertz's modification[5] of Willstaetter's method, using 5 grams of fresh leaves for each test. The carotin was

estimated colorimetrically. The petrol ether solution for the female Morus corresponded to a 0.09 per cent solution of potassium bichromate, while that from the male corresponded to a 0.07 per cent solution. The corresponding figures for Rumex are 0.07 per cent for the female and 0.04 for the male. In addition 4 other plants—2 of Morus (fresh leaves) and 2 of Cannabis (dried leaves)—were tested by the fractional adsorption method of Tsvett.[6] In these also, the females showed a visibly greater amount of carotin.

A large number of parallel tests with the Manoilov reaction were made with the same extracts used in the reduction reactions discussed above. Some of the plants used were the same as those tested a year previous and showed the constancy of the reactions in two growing seasons. Thus of 41 individuals of green plants retested, only 2 pairs (1 each in Morus and Populus) reacted differently from the year previous, and of 44 races of Mucors, only 2 pairs (in Rhizopus) gave a different reaction. In addition, 87 new individuals (79 green plants including 5 new genera and species, and 8 Mucors including 2 new genera and species) extended the number of plants tested with the Manoilov reaction. In all we have tested by this reaction 251 green plants (included in 24 species and 19 genera) in the same relative stage of development. Of the Mucors, we have tested 116 different races included in 20 species and 10 genera.

The results given in tables 1, 2, 3 confirm, in general, those reported in our preceding paper. A high average difference between female and male plants was obtained when the material was strictly comparable, the test succeeding with the Manoilov reaction in about 85 per cent of the cases. Extreme care is needed to have the material comparable in respect to factors affecting the extracts, such as the age of leaves, the conditions of light and moisture under which they have developed, etc. Strongly reacting extracts may have to be diluted with 60 per cent alcohol. Extracts which give a weak color reaction often may be readily distinguished by the brown sediment produced after the reagents are added. In female extracts, the sediment is more abundant, darker and disappears more slowly than in male extracts.

Comparison of the results obtained with the reduction and with the Manoilov reaction shows how closely the two reactions correspond (Tables 1, 2, 3). It is in but few cases that they were reversed or that one of them gave an inconclusive(?) result while the other worked distinctly. A sufficiently large number of plants have been tested to lead us to believe that the reduction reaction is nearly as useful as the Manoilov reaction when employed as a rough test for sex. At the same time it has the advantage of being a relatively simple chemical reaction requiring only one reagent and is a test in which the relative differences in strength of reaction in a given pair may be expressed quantitatively.

In a series of papers which have appeared recently (Schratz,[7] Alsterberg and Håkansson,[8] Riddle and Reinhart[9] and others), the Manoilov reaction is discussed from the chemical standpoint. These papers agree that the Manoilov reaction is a reduction phenomenon and that the final results depend on the quantity of various reducing substances which the tested material contains, although they differ somewhat as to the significance of the reaction.

Tests which we have made by adding various chemicals to 60 per cent alcohol in place of the plant extracts show clearly the correctness of this conception. Thus tartaric, oxalic, sulphuric and other acids tend to decolorize the solutions, while reducing substances such as dextrose, $FeSO_4$ and Na_2SO_3 leave the solutions more or less colored according to the concentration of the substances added, the effects therefore of these reducing substances corresponding to female reactions.

Hydrogen peroxide may apparently act both as an oxidizing and as a reducing agent[10] according to the concentrations employed. Thus 3 drops of a 3 per cent solution of purified H_2O_2 added to 3 cc. of 60 per cent alcohol left the solution weakly colored (a C reaction) while the same amount of a 6 per cent solution left the solution strongly colored (an A reaction). If 12 per cent or stronger solutions of H_2O_2 are used, the solutions are entirely decolorized and the reaction is like that of a male extract.

It is not our purpose at the present time to discuss the Manoilov reaction in detail from the chemical standpoint. We shall only mention here that our study of these reactions indicates that the reducing substances responsible for the differential behavior of extracts in respect to sex with both the Manoilov and the $KMnO_4$ reactions are probably not quite identical. The reduction of $KMnO_4$ will succeed under conditions under which the Manoilov reaction will not. Thus preparing the extracts with 95 per cent alcohol, or boiling them, or using ground material and centrifuged water extracts will give good results with $KMnO_4$ while the tests will be unsuccessful under these circumstances with the Manoilov reaction.

So far as other factors have been tested, both reactions behave similarly. Obviously both depend immediately on quantitative rather than qualitative differences in the extracts—changes in the concentrations altering or even reversing the reactions. It must be realized, however, that we are ignorant of the nature of the factors back of the biochemical end products determined by our tests.

The similarity of both reactions was shown further by the influence of various physiological conditions of the plants tested. The great majority of the tests reported were carried on with green plants, for the most part after they had flowered, and with Mucors after 7–10 days of growth.

The stage of development has been found to be a factor of great im-

portance, especially in the $KMnO_4$ reaction. Tests were repeated with extracts of leaves from the same plant: (*a*) when in the expanding bud, (*b*) at time of flowering, (*c*) 3–4 weeks after flowering, (*d*) in old yellowish leaves. Leaves collected 3–4 weeks after flowering from female plants which were fruiting and from male plants from which the flowers had fallen gave the most distinct differences between male and female individuals (88 per cent correct answers) and showed the fewest reversals, while leaves during the flowering period showed the least differences with the most reversals. In the latter case, as is seen in table 4, the percentage of correct identifications in regard to the $KMnO_4$ reaction drops to 65 per cent. The numbers obtained for the Manoilov reaction were 90 per cent and 67 per cent with the same individuals. The old yellowish leaves in both sexes gave mostly a weak female reaction, the female plants being still a little stronger.

TABLE 4

GREEN PLANTS. REDUCTIONS OF $KMnO_4$ AT DIFFERENT STAGES OF DEVELOPMENT

		BUDS STRONGER			FLOWERING STRONGER			SUMMER STRONGER			FALL STRONGER	
SPECIES	PLANTS	♀	♂	(?)	♀	♂	(?)	♀	♂	(?)	♀	♂
Rhamnus	10	2	1	2	2	3	0	4	1	0	3	2
Rhamnus	14	–	–	–	2	2	3	4	2	1	–	–
Morus	10	–	–	–	2	3	0	5	0	0	3	2
Populus	8	–	–	–	2	2	0	3	1	0	2	2
Rhus copalina	6	3	0	0	2	1	0	3	0	0	–	–
Salix	4	2	0	0	2	0	0	2	0	0	–	–
Ailanthus	2	1	0	0	1	0	0	1	0	0	–	–
Rumex	8	4	0	0	4	0	0	4	0	0	–	–
Myrica	8	–	–	–	3	1	0	4	0	0	–	–
Smilax	4	–	–	–	2	0	0	2	0	0	–	–
Celastrus	2	–	–	–	1	0	0	1	0	0	–	–
Sassafras	10	–	–	–	5	0	0	5	0	0	–	–
TOTAL	86	–	–	–	28	12	3	38	4	1	–	–
PERCENTAGES	—	–	–	–	65.1	27.9	7	88.4	9.3	2.3	–	–

As to the Mucors, well-developed cultures, 7–10 days old, were the best in comparison with young cultures 4–5 days or with old cultures 3–4 weeks old.

Other factors are being investigated which may have an influence on the reactions. Among these are the removal of flowers, the relative abundance of male and female flowers and of fruits on the plants tested.

Summary.—1. The power of plant extracts to reduce $KMnO_4$, tested in a large number of individuals, shows a high average difference between the sexes, the female plants being the stronger reducers.

2. A number of plants from last year's series were retested and tests of individuals from new species and genera confirm the results previously obtained in regard to the Manoilov reaction. A high average difference between sexes is obtained if the material tested is strictly comparable.

3. Both the $KMnO_4$ and Manoilov reactions run closely parallel.

4. The reducing substances responsible for the differential behavior in respect to sex of these two reactions, in part at least, are not identical.

5. Both reactions depend immediately on quantitative rather than qualitative differences.

6. The stage of development of the plant is a factor of importance. The most distinct differences found in green plants between male and female individuals were 3–4 weeks after flowering, while leaves collected during the flowering period showed the least differences. For Mucors the best results were obtained with cultures 7–10 days old.

[1] Paper presented before the Physiol. Sect. of the Bot. Soc. of America, December 29, 1926. This investigation was carried on under joint support of the Carnegie Institution of Washington and the Committee of the National Research Council for Research on Sex Problems to whom the authors gratefully acknowledge their indebtedness.

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INHERITANCE OF TRIMORPHISM IN LYTHRUM SALICARIA

BY E. M. EAST

HARVARD UNIVERSITY

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Numerous crosses between the three types of flowers found in *Lythrum salicaria* have given adequate basis for the following conclusions:

(1). The self-sterility of the species is not interpretable on the selective pollen-tube growth hypothesis found applicable to Nicotiana, Antirrhinum and Veronica.

(2). The two kinds of pollen in each form carry the same hereditary factors so far as the inheritance of trimorphism is concerned. Illegitimate pollinations with pollen from anthers not at the same level give the same results as legitimate pollinations with pollen from anthers at the same level.

(3). The flower differences are conditioned by three genes. Long-styled flowers are the ultimate recessive. Short-styled flowers are determined by a factor A, and may or may not carry Mid. Mid-styled flowers are conditioned by duplicate factors M_a and M_b in the same linkage

group. There is about 10 per cent crossing-over in both the male and the female gametes, though there appears to be slightly less crossing-over in the females than there is in the males. The presence of either or both of the factors conditioning Mid in the homozygous condition produces a lethal effect.

I have never obtained progeny from selfed Shorts, but all three forms have been obtained by other workers. Numbers were too small, however, to give dependable ratios. Longs produce only Longs. Mids presumably of constitution $\frac{M_a M_b}{m_a m_b}$, when selfed, yielded $75L$:$194M$. From this ratio it is clear that the cross-over percentage is about 10. The Mids produced will on theory be 57.6 per cent of type $\frac{M_a M_b}{m_a m_b}$, 6.4 per cent of each of the two types $\frac{M_a m_b}{m_a m_b}$ and $\frac{m_a M_b}{m_a m_b}$ and 0.71 per cent of the type $\frac{M_a m_b}{m_a M_b}$.

A series of Mids of these constitutions, presumably in somewhere near the proportion given above, were crossed with a Mid of constitution $\frac{M_a m_b}{m_a M_b}$. The result in 17 families was a ratio of $3L$: $117M$. A similar lot of Mids, though probably not having the same constitutions in the same proportion, were crossed with another Mid of constitution $\frac{M_a m_b}{m_a M_b}$. In a series of 31 families the resulting ratio was $15L$: $507M$. There was just under 3 per cent Longs in each case. Theoretically, when Mids of constitution $\frac{M_a m_b}{m_a M_b}$ are crossed (1) with Mids of constitution $\frac{M_a M_b}{m_a m_b}$ the result should be 4.3 per cent Longs, (2) with Mids of constitution $\frac{M_a m_b}{m_a m_b}$ or $\frac{m_a M_b}{m_a m_b}$ the result should be 3.3 per cent Longs, and (3) with Mids of constitution $\frac{M_a m_b}{m_a M_b}$ the results should be 0.5 per cent Longs. A mixture of these four types of crosses might well be expected to give 3 per cent Longs.

When Longs are crossed with Mids there should be two very different ratios. Mids $\frac{M_a m_b}{m_a m_b}$ and $\frac{m_a M_b}{m_a m_b}$ should give 50 per cent Longs; Mids $\frac{M_a M_b}{m_a m_b}$ should give 45 per cent Longs. We obtained 53.3 per cent Longs with 491 plants. The other ratio is when a Mid of constitution $\frac{M_a m_b}{m_a M_b}$ is crossed with a Long. Here 5 per cent Longs is expected. We obtained 8 per cent Longs in a total of 635 plants.

Longs crossed with Shorts which do not carry Mid should give $1L:1S$ We obtained $437L : 481S$.

Longs crossed with Shorts which carry Mid should also give two ratios that are similar and one that is wholly different from the other two. A $25L : 25M : 50S$ ratio is given with Short $\frac{A}{a}\frac{M_a m_b}{m_a m_b}$ or $\frac{A}{a}\frac{m_a M_b}{m_a m_b}$; a $22.5 L : 27.5M : 50S$ is given with Short $\frac{A}{a}\frac{M_a M_b}{m_a m_b}$. Presumably the second type is the commoner one. In such crosses we obtained $95L : 115M : 225S$—a percentage ratio of $21.8L : 26.4M : 51.8S$. The other type of Short, $\frac{A}{a}\frac{M_a m_b}{m_a M_b}$, when crossed with Long, gives a ratio of $2.5L : 47.5M : 50S$. In such crosses we obtained $9L : 77M : 116S$—a percentage ratio of $4.4L : 38.1M : 57.5S$.

Crosses between Mids other than those of constitution $\frac{M_a m_b}{m_a M_b}$ with Shorts *known not to carry Mid*, give ratios approaching $25L : 25M : 50S$. Slight departures decrease the Longs and increase the Mids. In a series of crosses our results were $126L : 125M : 243S$.

Crosses between Mids of constitution $\frac{M_a m_b}{m_a M_b}$ with *Shorts known not to carry Mid*, give the very different ratio of $2.5L:47.5M:50S$. We obtained $5.8L:41.9M:52.3S$, in a total of 86 plants.

Crosses between Mids and Shorts known to carry Mid can be of various types. We will report only one of our experiments here because of its critical value. Two crosses were made between Mids and Shorts in which the parental plants came from the same families. The crosses were made in such a way that, according to the simple two-factor hypothesis used by earlier writers for the interpretation of trimorphism, they should be duplicates of each other. Yet they gave quite different results. The first cross, made reciprocally, yielded $4L:87M:134S$, a ratio of $1.8L: 38.7M : 59.5S$. Now this is a reasonable fit either for a cross $\frac{a}{a}\frac{M_a M_b}{m_a m_b} \times \frac{A}{a}\frac{M_a m_b}{m_a M_b}$, or for a cross $\frac{a}{a}\frac{M_a m_b}{m_a M_b} \times \frac{A}{a}\frac{M_a M_b}{m_a m_b}$, both of which yield $2.2L : 47.8M : 50S$. Assuming that we are dealing with a cross of the first type, another Short parent from the same family may be a cross-over Short of constitution $\frac{A}{a}\frac{M_a M_b}{m_a m_b}$. Such a Short crossed with the same Mid or a Mid of constitution identical with that used in the first cross should give a ratio of $14.4L:35.6M:50S$. What we actually obtained was a ratio of $20L:54M:71S$, which is a percentage ratio of $13.8L:37.2M:49.0S$.

THE FASTING OF LARGE RUMINANTS

BY F. G. BENEDICT AND E. G. RITZMAN

NUTRITION LABORATORY OF THE CARNEGIE INSTITUTION OF WASHINGTON, BOSTON, MASSACHUSETTS, AND THE NEW HAMPSHIRE AGRICULTURAL EXPERIMENT STATION, DURHAM, NEW HAMPSHIRE

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The complete withdrawal of food from the living organism immediately throws the demand for energy to maintain life upon the organism itself, and profound changes may be expected in the adaptation of the vital activities to this new situation. Life itself now becomes part of a defensive scheme, with readjustments to secure a minimum demand upon the existing stores of energy and a prolongation of the draft upon organized body tissues. With cold-blooded animals fasting is a regular occurrence in the life cycle. The demand for energy is low, the reserves are usually well filled, and hence renewal of the store of energy is necessary only at long intervals. Warm-blooded animals, whose reserves are frequently lower and whose energy needs are greater, withstand fasting much less readily, for not only may their reserves be quickly depleted but the organized tissues as well may be attacked. Omnivorous or carnivorous animals during normal feeding have, as a rule, deposits of body fat, but because of the rapid digestion and absorption in the alimentary tract of these animals, these reserves are quickly exhausted when food is withheld. With ruminants the food residues in the intestinal tract are very large. Indeed, these residues in the case of the steer may amount at times to one-fifth of the entire body-weight. The possibility of these residues serving as a source of energy for a considerable period has been long assumed. Numerous fasting experiments of long duration (with men lasting over thirty days and with dogs over twice this time) have shown that the contents of the alimentary tract of man and the dog are almost immediately exhausted, and the whole expense of the life processes falls upon the deposits in the body.

The metabolism of the fasting ruminant has been but little understood, although a remarkable study on this subject was made in 1861 and 1862 by Hubert Grouven,[1] a study completely ignored by most workers in nutrition for over sixty years. To supplement the data of Grouven and to study by modern methods the influence of fasting upon the metabolism of ruminants, two adult steers and two young steers were subjected to fasting at the Laboratory for Animal Nutrition at the New Hampshire Agricultural Experiment Station, Durham, N. H., for from five to fourteen days at various times between December, 1921, and May, 1925. In each experiment daily records were made of the changes in body-weight, the amounts of drinking water, the volumes of urine and the weights of feces voided, the insensible perspiration, the heart-rate, the rectal temperature and the

muscular activity. A few records were also obtained of the respiration rate. In addition, observations were made of the number of urinations and defecations per 24 hours. The total loss of nitrogen in the urine was determined and a complete chemical analysis of the urine was made, as well as determinations of the nitrogen, dry matter, and water of feces. The particular feature of the experiments was the measurement of the gaseous metabolism, including the total carbon-dioxide production and, in all but the first two fasts, a determination of the respiratory quotient. From these two measurements the heat-production was computed for every experiment. It was deemed important to note the changes in these factors as the fast progressed, the influence of the state of nutrition prior to the fasting, and the course of the body demands at the different nutritive levels. The age factor, the influence of environmental temperature and the immediate reaction to the ingestion of food were supplementary problems.

The greatest stress was laid upon the measurements of the gaseous metabolism in a respiration chamber permitting the determination of the total carbon-dioxide production in periods as short as 30 minutes, in which case 4 consecutive half-hour periods usually sufficed. In addition, certain experiments were made in which the steer remained for three consecutive 24-hour periods inside the chamber. These experiments gave comparative values for the total 24-hour metabolism on the first three days of fasting.

The general behavior of the fasting steers led to the general deduction that the so-called "hunger feeling" is merely the temporary sensation caused by the physical contraction of the alimentary tract to meet the requirements of a diminished bulk, but in no sense represents distress due to a lack of nourishment of the tissues.

The gross changes in body-weight showed the expected daily losses resulting in great part from the loss of urine and particularly of feces. The character and the amount of food prior to the fast affected these daily losses. Thus, the losses were largest after pasturage, less after maintenance feeding and smallest after a period of submaintenance feeding. It was clearly shown that live weight losses were no indication of real tissue changes. The latter can be intelligently studied only by urine analyses and particularly by respiration experiments, which give the data for calculating the total loss of carbon and, in turn, the total heat output.

Urine Analyses.—From the many urine analyses made by our associate, Dr. T. M. Carpenter, the urinary nitrogen per day has been chosen for presentation in table 1, as indicative of the protein disintegration. In this table the initial level of the nitrogen output, the changes during the first few days, and particularly the approximate level at the end of the fast are of special interest. Large differences in the nitrogen level are usually accounted for by the differences in the previous food level and particularly in the state of nutrition.

TABLE 1

DAILY NITROGEN EXCRETION (IN GRAMS) IN URINE OF FASTING STEERS

STEER AND DATES OF FASTS	COMPOSITE SAMPLE ON FEED BEFORE FAST	DAYS FASTING 1	2	3	4	5	6	7	8	9	10	11	12	13	14
Steer *C*															
Dec., 1921	59.4	64.7	60.0	81.5	42.3	50.8	55.5	42.7	....	....	....	....	....	....	....
Jan., 1922	95.5	123.0	118.0	89.8	44.2	66.9	62.3	37.1	43.6	30.4	44.7	....	....	....	....
Apr., 1922	72.3	78.2	75.2	67.8	78.6	57.9	63.9	42.9	48.8	52.3	34.3	51.9	36.3	39.7	47.2
June, 1922	80.0	100.0	103.0	80.5	62.0	60.8	....	....	....	....	....	....	....	....	....
Nov., 1922	....	99.0	100.6	59.4	94.5	80.0	49.3	73.4	66.5	72.0	....	....	....	....	....
Nov., 1923	....	....	87.0	95.4	78.1	63.6	....	....	....	. ..					
Mar., 1924	....	15.6[1]		102.9[2]		40.2	37.3	40.4	38.7	39.0	29.7	....	....	. ..	....
Steer *D*															
Dec., 1921	57.5	49.3	50.7	48.3	61.2	32.7[3]	53.1	50.7	....	....	....	....	....	....	....
Jan., 1922	101.5	122.0	89.9	85.4	66.8	57.0	49.9	55.7	37.8	32.2	35.6	....	....	....	....
Apr., 1922	76.7	91.1	88.9	78.7	71.1	62.3	56.5	38.9	44.0	38.3	45.1	35.0	37.3	35.8	33.9
June, 1922	82.7	93.8	96.2	88.1	66.1	62.9	....	....	....	....	....	....	....	....	....
Nov., 1922	....	99.4	91.6	93.3	80.0	86.0	71.3	47.8[4]	54.5	....	....	....	....	....	....
Nov., 1923	....	....	81.0	91.7	80.5	73.3	....	....	....	....	....	....	....	....	....
Mar., 1924	....	18.8[1]	31.2	34.1	35.8	51.2	42.5	45.4	37.1	41.1	....	....	....	....	....
Steer *E*															
Feb., 1924	19.8	13.8	24.6	22.2	36.4	26.6	....	....	....	....	....	....	....	....	....
Steer *F*															
Feb., 1924	20.6	10.8[5]	20.2	22.2	32.0	35.5	32.0	....	....	....	....	....	....	....	....

[1] The period of collection was for only 17 hours.

[2] Data for March 4 to 7 combined, because analyses of urine could not be made in exact 24-hour periods during this time.

[3] Some urine lost; amount not known.

[4] Approximately 200 gm. urine lost.

[5] About 300 grams of urine lost during the day.

With men a rise in the nitrogen output during the first three days of fasting has been shown to be directly correlated with an increased draft upon body glycogen. With ruminants, although the nitrogen output on the first day of fasting is usually greater than on the last day when food is eaten, there is no characteristic course of the nitrogen output indicating other than a general tendency to fall off as the fast progresses. At the end of the longer fasts (10 to 14 days) the adult steers, *C* and *D* (weighing circa 600 kg.), excreted about 40 to 50 gm. of nitrogen per day. In the March, 1924, fast, which followed a long period of submaintenance feeding (when there were heavy drafts upon the body nitrogen before the fast began), the initial loss was nearer 30 gm. per day and the output rose so that on the last three days of the fast it averaged nearer 40 gm. per day. This finding after submaintenance feeding was likewise confirmed with the younger steers, *E* and *F* (weighing circa 350 kg.), whose daily nitrogen excretion was about 15 gm. on the first day of fasting after submaintenance feeding and increased to about 35 gm. as the fast progressed.

The detailed chemical analyses of the urine showed that as a result of fasting the composition of the urine changed from that characteristic of herbivora when on feed (with a low content of urea and a relatively high content of hippuric-acid and amino-acid nitrogen) to that characteristic of the urine of a fasting man (with a high content of urea nitrogen and a low content of hippuric-acid nitrogen and amino-acid nitrogen). The low ammonia and the low ketone bodies showed that acidosis—a common feature with fasting omnivora—was absent.

The Total Gaseous Metabolism and the Computed Heat-Production.—The nitrogen loss indicates approximately the drafts upon the body protein, but for sustaining life processes during fasting by far the largest draft upon the body stores is made upon fat, and for a study of such losses the measurement of the gaseous exchange is essential. The carbon dioxide resulting from the oxidation of food or body material is a rough index of the total heat-production. Since, however, carbon dioxide resulting from the combustion of fat is accompanied by about 30 per cent more heat than when the carbon dioxide results from the combustion of carbohydrate, there will always be a large inherent error in using the carbon-dioxide production alone as an index of the heat-production without a knowledge of the exact character of the combustion, i.e., under conditions where there may be a combustion predominatingly of carbohydrate or predominatingly of fat. Fortunately, in fasting experiments the metabolism rapidly reaches a condition where the combustion is predominatingly fat, so that after the first day or two carbon-dioxide measurements have an especially high value in indicating the intensity of the metabolism. If one can determine the oxygen consumption, the computation of the energy produced is much more exact, for when one liter of oxygen is absorbed in the body, the heat evolved

is essentially the same whether the body is burning fat or carbohydrate, the extreme difference being but 6 per cent. If, in lieu of determining the oxygen consumption directly, one can determine the total carbon-dioxide output and simultaneously the respiratory quotient, an extremely accurate calculation of the heat-production may be made. In our earlier studies the carbon-dioxide production only was determined. Subsequently, thanks to Dr. T. M. Carpenter's gas-analysis apparatus,[2] we were able to determine the respiratory quotient in numerous experiments covering the first few days of fasting, thus obtaining a clear picture of the changes to be expected in the character of the metabolism as the fast progresses, so that we may legitimately apply standard respiratory quotients to much of the earlier data. The calorific value of carbon dioxide, that is, the amount of heat produced in the production of one liter of carbon dioxide, is definitely known for various respiratory quotients, and we have computed the heat-production of our large steers upon this basis.

In order to have the experiments comparable from day to day and to note the differences between different animals, particularly between the adult and the younger animals, some standard condition of measurement had to be prescribed. For our "standard metabolism" test we insisted that the animal should have been at least 24 hours without food, that there should be a minimum degree of muscular activity during the measurements, and that the animal should always be standing. Throughout all of the fasting experiments the metabolism was measured in the respiration chamber in numerous consecutive half-hour periods. From these measurements and from the respiratory quotient the heat-production was calculated. Obviously, as the fast continued and the animal lost weight (i.e., became smaller), the heat-production would automatically become somewhat less. In the attempt to compare all four steers on the same basis, we have computed the heat-production for every day per 500 kg. of body-weight (the average weight of an adult steer) and likewise per square meter of body surface. This latter computation is looked upon by many physiologists as of exceptional significance in equalizing the values for the heat-production not only of all ruminants but of all warm-blooded animals. With this view we are not in accord, but our results are reported in table 2 upon this basis of comparison as a concession to physiologists who hold this view. (Surface $= W^{2/3} \times 0.1081$; W = live weight in kg.)

The data in table 2 show that there is a distinct tendency for the heat-production per square meter of body surface to fall off rather rapidly during the first one or two days of fasting. There are, however, pronounced differences in the course of the metabolism in the different fasts, which can be interpreted only by taking into consideration the previous state of nutrition and the character of the previous feed. Thus, prior to the first four fasts of steers *C* and *D* a maintenance ration of hay and meal

TABLE 2

HEAT-PRODUCTION PER SQUARE METER OF BODY SURFACE PER 24 HOURS DURING FASTS OF 5 TO 14 DAYS

STEER AND DATES OF FASTS	DAYS FASTING 1	2	3	4	5	6	7	8	9	10	11	12	13	14
	HOURS WITHOUT FOOD 22–32	42–45	65–80	89–104	113–128	137–149	168–174	192–200	216–228	241–245	267–272	291–300	312–315	336–339
	CAL.	CAL.	CAL.	CAL.	CAL.	CAL.	CAL.	CAL.	CAL.	CAL.	CAL.	CAL.	CAL.	CAL.
Steer *C*														
Dec., 1921	1880	1680	1560	1520	1500	1490	1480	....	....	....	....	....	....	....
Jan., 1922	1960	1890	1690	1500	1600	1510	1520	1400	1510	1590	....	....	....	....
Apr., 1922	1800	1470	1530	1450	1420	1500	1410	1500	1510	1410	1360	1400	1340	1300
June, 1922	1920	1690	1460	1530	....	1580	....	....	....	....	....	....	....	....
Nov., 1922	2090[1]	1780[1]	1690[1]	1520[1]	1550[1]	....	....	1590	1680	....	....	....	....	....
Nov., 1923	....[2]	1880	1670	1500	1610	1670	....	....	....	....	....	....	....	....
Mar., 1924	1290	1110	1180	1220	1230	1200	1270	1240	1230	1150	....	....	....	....
Steer *D*														
Dec., 1921	2080	1680	1690	1620	1600	1550	1530	....	....	....	....	....	....	....
Jan., 1922	2060	1760	1630	1560	1570	1500	1530	1420	1460	1470	....	....	....	....
Apr., 1922	1740	1630	1560	1440	1430	1410	1410	1420	1410	1360	1460	1390	1390	1400
June, 1922	2130	1830	1810	1620	1740	....	....	....	....	....	....	....	....	....
Nov., 1922	....	2060	1860	1840	....	1670	1600	....	....	....	....	....	....	....
Nov., 1923	2490	2090	1830	1790	1650	....	....	....	....	....	....	....	....	....
Mar., 1924	1770	1580	1490	1470	1380	1560	1430	1230	1260	....	....	....	....	....
Steer *E*														
Feb., 1924	1600	1690	1550	1590	....	....	....	....	....	....	....	....	....	....
Steer *F*														
Feb., 1924	1560	1620	1700	....	1570	....	....	....	....	....	....	....	....	....

[1] Measurement made during period of lying and standing.

[2] A 24-hour heat-production of 2480 calories per square meter of body surface was noted when the steer had been 18 hours without food.

had been fed; prior to the fasts in November, 1922 and 1923, steers *C* and *D* had been upon pasture, and prior to the fasts in March, 1924, they had been for 2 months upon a submaintenance ration of hay. Finally, it should be noted that steers *E* and *F* were much younger animals and that their fasts were preceded by a long period of submaintenance feeding on hay and meal. An examination of the data with these points in mind shows that there was usually a pronounced fall in the heat-production on the first and second days of the fast. Thereafter the metabolism decreased more slowly, reaching in the case of the 14-day fast a level of about 1300 or 1400 calories. In the fasts following pasturage in November, 1922 and 1923, the initial values are high, and usually the heat-production remains at a higher level as compared with the values for corresponding days in the other fasts. This may possibly imply that the steers were in a better nutritive condition after their long period of feeding on a good New Hampshire pasture than they were in the first four fasts, when they were fed presumably maintenance rations under stall conditions.

In the March fast, which followed submaintenance feeding, the initial level with steer *C* is low, actually a trifle lower even than it is at the end of the 14-day fast. There is not much change in the subsequent nine days. Steer *D*, on the other hand, begins this fast at a much higher level, and there is a tendency for the values to decrease. Indeed, the low values on the eighth and ninth days, which are materially lower than on the corresponding days in any other fast, are probably accounted for by the low nutritive level at which this animal began to fast.

It is especially to be noted that with the younger steers, even following submaintenance feeding, the metabolism on the several fasting days is, in general, higher than the metabolism noted after submaintenance feeding with steer *D* and notably higher than that of steer *C*. This high level with these two animals may be taken as an index of the higher metabolism of the more immature protoplasm.

Since these observations were made when the animals were standing, it is not justifiable to compare these results directly with the "basal metabolism" of other animals, as they do not represent what might be called the basal metabolism of steers. This feature will be considered in a following paper.

The details of these experiments with fasting steers are about to appear as a monograph from the Carnegie Institution of Washington, Publication No. 377, entitled "The Metabolism of the Fasting Steer."

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THE BASAL METABOLISM OF STEERS

BY F. G. BENEDICT AND E. G. RITZMAN

NUTRITION LABORATORY OF THE CARNEGIE INSTITUTION OF WASHINGTON, BOSTON, MASSACHUSETTS, AND THE NEW HAMPSHIRE AGRICULTURAL EXPERIMENT STATION, DURHAM, NEW HAMPSHIRE

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The character and the amount of food eaten, the time and the rapidity of eating and the muscular activity incidental to a 24-hour life are so variable with different animals and, indeed, with different humans, that physiologists have attempted to prescribe certain definite conditions of experimenting in order to rule out the effect of these variables upon metabolism and to measure the metabolism under conditions in which all external and, so far as possible, internal factors affecting metabolism are eliminated. The main object of measuring the basal metabolism is to determine what one may call the "over-head cost" of living of the organism, that is, the irreducible minimum. The measurement of the basal metabolism has further importance in comparative physiology, in that it is the only logical basis for the comparison of the relative heat-production of different individuals with each other or with different species.

With humans these basal measurements are made in the so-called "post-absorptive condition," which may usually be obtained 12 hours after the last meal, if the meal is not too rich in protein or calories. The subject should be lying, well covered with clothing, comfortable, psychically and muscularly in repose and obviously without fever. With animals comparable conditions are difficult to secure. In the case of the steer, for instance, food may be withheld, the body temperature may be normal, and muscular activity may for the most part be ruled out, and yet it is impossible to make a steer lie down and remain lying for any length of time. Moreover, thus far the element of protection from the external temperature, other than the normal coating of hair, has not been considered. Obviously extremes in temperature are to be avoided, for it has been found that with the rat and the dog small differences in temperature result in considerable differences in metabolism, induced either by restlessness, shivering or possibly a stimulus to greater heat-production to combat the effect of cold. With human beings, well covered as they are, temperature variations, unless extreme, play a small rôle. Observations thus far with the steer imply that the influence of environmental temperature is likewise small, particularly if the animal is normally fed. Indeed, an extensive series of experiments has shown that the temperature factor is much less pronounced with ruminants than with either the rat or the dog, and as practically all of our experiments with steers were made at or near 20°C. (the temperature

commonly assumed for man), this factor can be disregarded in the subsequent considerations.

One of the chief difficulties in determining the basal metabolism of the steer is the question as to when the post-absorptive condition is reached. With humans it is found that the appreciable rise in metabolism following (usually immediately) the ingestion of food disappears in about 12 hours. After this time the metabolism remains essentially constant for many hours, but as prolonged fasting continues the heat-production continually and regularly falls. In the case of ruminants it is important to find at what time after the withholding of food the metabolism reaches an approximate plateau, from which it descends only slowly with continued fasting. In the series of fasting experiments reported in a preceding article,[1] measurements were secured in 24-hour units indicating how the fasting metabolism of the steer decreases with the length of the fast. Thus, in the long fasts the heat-production decreased rapidly on the second day of fasting. There was frequently a further decrease on the third day. In the fasts other than those following submaintenance feeding, twelve experiments showed that approximately a plateau in metabolism was reached on the second day in four cases, on the third day in six cases and on the fourth day in only two cases. Thus, the indications are that the plateau would occur during the period beginning between 65 and 80 hours after food.

A series of 4-day fasting experiments, in which the metabolism was measured in continuous 8-hour periods during the second, third and fourth days of fasting, gives us a better picture of the incidence of a plateau in the fasting metabolism. In these experiments it was found that the metabolism on the three successive days remained for the most part uniform, with a tendency for a plateau to have been reached about 32 hours after the last food was eaten. Judging from these four continuous 3-day respiration experiments, therefore, if one begins the experiment 24 hours after the last food and the previous ration has been a maintenance ration, the metabolism on the three successive days is essentially constant. A further series of experiments made with two other steers, measured in 8-hour periods, showed that following a maintenance feed-level the plateau was reached usually at about 24 to 30 hours after food.

From the well-known effects of prolonged fasting upon metabolism it is clear that one cannot consider the metabolism on the fourteenth day of fasting as the basal metabolism. On the other hand, on the first day of fasting there is unquestionably the after-effect of previously ingested food. It would seem more logical, therefore, to employ the measurements made about the thirtieth hour after food as corresponding to the 12-hour interval without food in the case of humans.

By suitable registration of the activity or, more properly speaking, the degree of muscular repose (i.e., by using the suspended platform and the

tambour and kymograph connections such as are used in all metabolism experiments made in connection with the Nutrition Laboratory's researches), periods of unusual activity can be immediately ruled out. There remains to be considered, therefore, the influence of the lying versus the standing position.

In order to determine the basal metabolism of the steer and have it strictly comparable with basal measurements obtained with man and, indeed, with dogs, rats, and other animals, one should make the measurements when the steer is in the greatest degree of repose, which, *a priori*, would be in the lying rather than in the standing position. Experiments with steers while lying are possible, but difficult to secure and represent only periodic fragments in long respiration experiments. Our respiration experiments were of two types: (1) those of four consecutive half-hour periods, when the animal was continuously standing, and (2) those consisting of three or four consecutive 24-hour periods, when the animal was allowed to stand or lie at will. Singularly enough, the metabolism as measured on both of these bases showed great uniformity. Thus, on the days when the animals were lying down for nearly half of the time, the total metabolism per 24 hours was essentially the same as when it was computed from a series of four half-hour periods when they were standing continuously. Fragmentary observations made during lying periods in the fasting experiments showed that at the beginning of the fasts there was a fairly pronounced difference of perhaps 20 to 30 per cent between the metabolism in the lying and in the standing position, but that this difference distinctly decreased as the fasts progressed. This seeming absence of difference between standing and lying has justified Professor Møllgaard[2] in disregarding any correction for body position.

Anatomists have expressed the opinion that the adjustment of the steer's leg muscles are such that the effort of standing is minimum. Since, however, it is important from the standpoint of comparative physiology to study the basal metabolism of various species and since it seems illogical to compare the basal metabolism of a lying man with that of a standing steer, we have made every effort to secure the most probable correction for lying. This correction may be as large as 30 per cent or as small as 9 per cent. The average metabolism of our steers, when standing, 42 to 56 hours after the last food (maintenance ration) was eaten, was approximately 1700 calories per square meter of body surface per 24 hours. If we deduct 30 per cent for the effort of standing, to bring the animals to the presumed lying level (and we believe this correction is fully as large as is permissible), the corrected value would then be 1200 calories. Since our steers were studied in all but two instances under stall confinement and they were by design a little under-nourished rather than over-nourished, both factors would make for a sub-normal metabolism. We believe, therefore,

that the basal metabolism of the steer is on the average much nearer 1300 calories. There is distinct evidence that it will be larger with the younger steers and will be reduced in those cases where a period of undernutrition precedes the "basal" measurement, but we believe that the true basal metabolism of steers will in general be 1300+ calories.

This value of 1300 calories is 35 per cent higher than the average value of 964 calories commonly ascribed to steers, when lying 24 hours, as the result of earlier values published by Professor Armsby and his associates.[3] In comparing these values, however, one should realize that the earlier values were computed by measuring the metabolism of the steer with a maintenance ration and then measuring it again after a period of several weeks' feeding with a submaintenance ration. The degree of the lowering in metabolism was ascribed to the effect of the curtailment of the ration, and the fasting metabolism was computed as a linear function of the difference in metabolism due to the difference in the amount of food ingested. From our studies on the effects of submaintenance rations and of undernutrition, we are convinced that in the above-outlined experiments the authors were measuring, in the second period, not simply the effect of food withdrawal but likewise the superimposed effect of a several weeks' period of submaintenance feeding, i.e., undernutrition.

In a study of the basal metabolism of humans[4] the Nutrition Laboratory found that the average basal heat-production of 136 men was 925 calories per square meter of body surface per 24 hours and that of 103 women was 850 calories. New-born infants have shown very low values, nearer 600 calories. Our observations on the white rat, as yet unpublished, have also given us values nearer 600 calories, in full support of the findings of Mitchell,[5] who reports values averaging about 600 calories. In this relatively small group of warm-blooded animals, therefore—the rat, man and the steer—there is a variability in the heat-production per square meter of body surface ranging from 600 to 1300 calories per 24 hours. It is hardly possible that this discrepancy can be explained by any serious errors in the determination of the surface area, for the Du Boises[6] have made the measurement of the surface area of man one of our most accurately known anthropometrical observations, and Moulton,[7] subsequently Hogan and Skouby,[8] and more recently Brody,[9] have made extensive contributions regarding the surface area of cattle. The Hogan formula was used in computing the surface areas of our steers. If the more recent formula of Brody had been employed, the areas would be on the average about 10 per cent lower and the metabolism per square meter of body surface would be correspondingly higher. It would, therefore, appear that, according to the present-day methods of measuring the basal metabolism and the surface area of the steer and man, the heat-production of the steer under conditions reasonably comparable to those obtaining during basal metabolism meas-

urements with men is 1300 calories per square meter of body surface on the average, i.e., a full 40 per cent greater than that of man, a fact completely at variance with the well-known "law of surface area." This conclusion is based, for the most part, on material published in detail elsewhere[10] and is wholly confirmed by actual measurements (as yet unpublished) of the metabolism of steers and cows, during periods when they were *lying* down and after the period of active digestion had passed.

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THE METABOLIC STIMULUS OF FOOD IN THE CASE OF STEERS

BY F. G. BENEDICT AND E. G. RITZMAN

NUTRITION LABORATORY OF THE CARNEGIE INSTITUTION OF WASHINGTON, BOSTON, MASSACHUSETTS, AND THE NEW HAMPSHIRE AGRICULTURAL EXPERIMENT STATION, DURHAM, NEW HAMPSHIRE

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Rubner, whose classical contributions to the science of nutrition have never been equaled, working chiefly with dogs, found that the increase in heat-production following the ingestion of food was greatest with protein, much less with fat and least with carbohydrate, and proposed the expression "specific dynamic action" to represent this effect of food. Zuntz, working about the same time with ruminants, noted large increases in the heat-production following the ingestion of food by these animals, in many instances even with rations relatively poor in protein, and he was inclined to attribute the increase to the "work of digestion," primarily to the mechanical effort of moving the large intestinal residues through the body, although he did not by any means rule out other incidental factors such as

glandular activity. In experiments made by us at the Laboratory for Animal Nutrition at the New Hampshire Agricultural Experiment Station, Durham, New Hampshire, with a respiration chamber in which fasting steers could be studied, certain data were obtained on the differences in the metabolic level of the steer with and without food. These data contribute to the mooted questions as to the magnitude of the increment in metabolism after the ingestion of food and as to the identity of certain factors influencing this magnitude, and we therefore believe it is justifiable to present or rather revive an hypothesis as to the cause of these increments.

Experiments were made in which the steer (weighing about 350 or 400 kg.) was placed inside a respiration chamber for four consecutive days, during all of which time the gaseous metabolism was measured. The ration to be studied had been given for several weeks prior to the test, and during the first two days inside the chamber the steer was fed exactly as usual. On the last two days, however, no food was given. The apparatus was frequently tested for accuracy, by admitting known weights of carbon dioxide from weighed bottles of the liquefied gas, and the amounts recovered by the apparatus rarely varied by more than $\pm$ 1.5 per cent from those admitted. The steer was allowed to stand or lie down at will, and usually spent a little more than one-half of the time during the twenty-four hours in lying. In certain experiments the ration studied consisted of timothy hay, containing 0.99 per cent of nitrogen on the air-dry basis. In other experiments alfalfa hay containing 2.06 per cent of nitrogen was fed. The experiments were all made at an environmental temperature of approximately 22°C. In some instances the daily ration consisted of 7 kg. of hay; in others it was reduced to 3.5 kg. The details of these experiments are about to appear in a monograph from the Carnegie Institution of Washington.[1]

The data secured for the two days with food and the second day without food are recorded in table 1. To avoid complicating the records, the intermediary value for the first day without food has been omitted. On this day the metabolism was much lower than it was on the two days with food, but was somewhat higher than on the second fasting day. On the second day it was found[2] that an approximate plateau is reached, which is only slowly lowered with prolonged fasting. We believe it justifiable, therefore, to consider that the metabolic level on the second day without food represents essentially the basal condition, comparable, at least so far as food intake is concerned, with that commonly prescribed in experiments with humans. Indeed, this belief is substantiated by a determination of the metabolism on the third day of fasting in the case of experiment No. VI with steer *F*, when the heat-production was found to be practically the same as on the second day, i.e., 7600 as compared with 7500 calories.

From table 1 it is seen that the metabolism on the two days with food was essentially constant. In experiment No. I with steer *E*, the total 24-hour heat-production during the two days when 7 kg. of timothy hay were fed

TABLE 1

METABOLIC STIMULUS OF TIMOTHY AND ALFALFA HAY

EXPERIMENT NO.	STEER	LIVE WEIGHT KG.	HOURS STANDING	HOURS LYING	HAY EATEN IN 24 HOURS	HOURS WITHOUT FOOD TO BEGINNING OF DAY	HEAT (COMPUTED) PER 24 HOURS TOTAL CAL.	PER CENT INCREASE DUE TO FOOD
		368.8	14	10	7 kg. timothy	0	11,800	
I	*E*	(368.8)	11	13	"	0	11,500	51
		349.4	7½	16½	None	32	7,700	
		435.2	9	15	7 kg. timothy	0	11,900	
II	*F*	(435.2)	11	13	"	0	11,900	47
		415.8	7½	16½	None	32	8,100	
		352.8	11	13	7 kg. alfalfa	0	11,600	
III	*E*	(352.8)	10½	13½	"	0	11,400	64
		339.0	8	16	None	32	7,000	
IV	*E*	362.8	13½	10½	7 kg. alfalfa	0	11,500	
		350.4	6½	17½	None	32	6,700	72
		423.2	13½	10½	7 kg. alfalfa	0	12,500	
V	*F*	(423.2)	10½	13½	"	0	12,600	63
		399.8	10½	13½	None	32	7,700	
		424.0	11	13	7 kg. alfalfa	0	12,300	
VI	*F*	(424.0)	11½	12½	"	0	12,300	64
		(410.0)	—	—	None	32	7,500[1]	
		351.4	13	11	3.5 kg. timothy	0	8,200	
VII	*E*	(351.4)	10½	13½	"	0	7,900	34
		333.4	12½	11½	None	48	6,000	
		408.6	14	10	3.5 kg. timothy	0	9,200	
VIII	*F*	(408.6)	10	14	"	0	8,800	38
		399.6	10½	13½	None	48	6,500	
		346.6	9½	14½	3.5 kg. alfalfa	0	7,800	
IX	*E*	(346.6)	9½	14½	"	0	7,700	41
		339.0	9½	14½	None	48	5,500	
		407.0	12½	11½	3.5 kg. alfalfa	0	9,100	
X	*F*	(407.0)	13	11	"	0	9,000	47
		392.6	11	13	None	48	6,200	

[1] On the following day, also without food, the metabolism was 7600 calories.

averaged 11,600 calories. On the second day without food it was 7700 calories. The difference, 3900 calories, represents an increment in metab-

olism above the basal value of 51 per cent due to the ingestion of the 7 kg. of timothy hay. In experiment No. II with steer *F* the ingestion of 7 kg. of timothy hay gave an average heat-production of 11,900 calories. The basal value was 8100 calories, and the increment due to the food was therefore 3800 calories or 47 per cent. In another series of experiments the 7 kg. of timothy hay were replaced by 7 kg. of alfalfa hay, and in the first of these experiments (i.e., experiment No. III with steer *E*) the average metabolism for the first two days was 11,500 calories, the basal metabolism was 7000 calories, and the increase was 4500 calories or 64 per cent. A duplicate experiment with this same animal under the same conditions gave an increment of 72 per cent. With steer *F* two experiments with 7 kg. of alfalfa hay gave increases of 63 and 64 per cent, respectively. The 7 kg. of either hay was designed to be slightly below rather than above the maintenance requirement.

In other experiments the ration was reduced by one-half or to 3.5 kg., and the steer was maintained upon this submaintenance ration for several weeks. These experiments showed clearly the effect of undernutrition in a profoundly lowered basal metabolism. A pronounced increment in metabolism during the feeding days as compared with the days without food nevertheless existed in all cases.

Since there is not a striking difference in the increases caused by the alfalfa and the timothy hay, in spite of the fact that alfalfa hay contains double the protein of timothy hay, we are forced to the conviction that we are dealing here with an increment which is essentially independent of the protein content of the ration, i.e., an increment determined by some other factor. Increments of this order of magnitude, 50 to 60 per cent, are commonly found with humans, and, indeed, with dogs only after feeding protein-rich rations and then generally for but short periods. With the ruminant, therefore, it must be recognized that a protein-poor ration, approximately maintenance, produces an enormous increase in the heat-production above a reasonably assumed basal value.

Part of this increment may logically be ascribed to the fact that there was somewhat more physical activity inside the respiration chamber on the days when food was given, due to the actual eating of the food. Moreover, there were differences in the time spent standing and lying, the steer, in general, lying a somewhat longer period of time on the days without food. Exact records, however, show that this difference usually amounts in any one series of experiments to hardly more than one or at the most two hours. Nevertheless, a correction should, theoretically, be made for the difference in activity, before comparing the days on food with the day without food. The magnitude of this correction is debatable. An inspection of our data shows that it can hardly be 5 per cent in general. Ideal conditions would call for carefully studied periods in which the animal was lying during both

the food and the basal periods, but as yet these have not been secured. It seems incredible, however, that an increment in the heat-production of the order of magnitude here shown could be to any degree wiped out by any correction for differences in muscular activity.

A second possible explanation of this increment is in the earlier theory of Zuntz, who maintained that the movements of large masses of food through the intestinal tract called for a "work of digestion" to not a small degree. Experiments made in the Nutrition Laboratory on humans, to whom purgatives and agar-agar were given,[3] and on dogs with deficient absorption and, hence, enormous stools,[4] point to an insignificant effort (expressed in terms of energy) involved in the passage of fecal material through the intestinal tract.

The most probable explanation of these increments in heat-production found with a protein-poor ration, such as timothy hay, is, we believe, the old theory of Grouven,[5] so long neglected and, indeed, based upon wholly insufficient evidence. Grouven maintained that carbohydrates are not absorbed by the ruminant as glucose, but are transformed by fermentation in the large intestinal tract into the lower fatty acids and are wholly absorbed in the fatty-acid stage. We believe that the absorption through this path pours into the blood a large amount of fatty acid which, carried to the cells, stimulates their activity. This statement is made in spite of the fact that even during fasting the urine of the steer shows almost entire absence of the normal indication of acidosis in humans, i.e., acetone and β-oxybutyric acid. Although a stimulus theory was put forth a number of years ago by Friedrich Müller,[6] and although in a discussion of this point at the Fifteenth International Congress on Hygiene[7] it was further elaborated that probably these bodies were of an acid nature, it is only just to note that in a recent personal conversation, Professor Müller states that he today disavows this belief. It is our conviction, however, that the evidence secured with these steers leaves the supposition of an acid stimulus as the most plausible hypothesis to explain the greatly increased metabolism following the ingestion of a protein-poor ration.

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STUDIES ON THE EFFECT OF POTASSIUM ALUM-HYDROCHLORIC ACID SOLUTIONS ON THE GROWTH AND FATE OF NEOPLASTIC TISSUE

1. EFFECT ON A SLOW-GROWING ADENO-CARCINOMA OF THE MOUSE

BY LEONELL C. STRONG

SCHOOL OF PUBLIC HEALTH AND BUSSEY INSTITUTION, HARVARD UNIVERSITY

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For the past nine years I have been engaged in the study of the genetic phenomena correlated with the incidence, growth and ultimate fate of spontaneous as well as transplantable neoplasms.

My experimental animals have been carefully selected and pedigreed from more than thirty generations. At the present time, I have stocks of mice that have given consistent results on transplantation for several years. The normal behavior and growth of the neoplastic tissue during the process of transplantation has been determined.

At the same time that these experiments on transplantation have been carried on, I have developed, by selective matings, a stock of mice of which the females develop spontaneous mammary gland carcinoma—provided the individuals be used as breeders. This spontaneous neoplastic tissue is usually multiple in origin and very frequently metastasizes to the lungs. A criticism with reference to these stocks with spontaneous tumors has been recently made, it being maintained that these neoplasms are *induced*. To any one who has worked with spontaneous mammary gland carcinomata as they arise in mice, the fallacy of this criticism is apparent. I have used no irritant or other means to induce these neoplasms. That they are not induced but that they arise spontaneously is borne out by the fact that other stocks of animals free from spontaneous cancer are constantly kept under the same conditions as the tumor-bearing stocks. As far as can be determined, the incidence of spontaneous mammary gland carcinoma has been brought about by the functioning of hereditary predisposition factors that responds to the genetic process of selection. The only other factor involved as far as can be determined is the fact that the females must be used as breeders.

We are, therefore, in a position to continue experimentation on these genetically determined stocks in order to ascertain, if possible, the environmental or secondary factors that call forth this hereditary predisposition factor toward the spontaneous production of neoplasms. Any chemical, physical or other agent that will influence the rate of growth of transplanted or spontaneous tumors so as to bring about either the retardation of their growth or their complete or partial regression may throw some light on this complicated biological problem.

With this aim in view, I have tried out on tumor-bearing mice a good many chemicals and combinations of chemicals, introducing them into the organism either by mouth (in the drinking water or food) or by injection subcutaneously. It was suggested to me that perhaps potassium alum would affect the growth of the tissue. I have used this chemical in a solution of hydrochloric acid and distilled water with promising results.

This paper is intended as a preliminary report in order to call the attention of other investigators to the data at hand. It is hoped that confirmatory results may be obtained on comparable material. It is my purpose to publish the results obtained by variations in technique in a series of short papers within the near future.

Method.—The technique of administering the material by mouth is not easy. Due to the extreme bitterness of the mixture, the experimental animals do not readily take the desired dosage in their drinking water. Mice can get along on very little water, and many of the animals take considerably less of the water to which the alum solution has been added than of pure water. This decrease of water intake interferes with the growth of the normal animal and consequently has complicated the results obtained in the early work. This difficulty can be partly overcome by giving fresh water after the required dosage has been taken, by giving fresh water every seventh day or by the use of the second method of administration mentioned below.

Several methods of administering the material in the drinking water have been used. The earlier method was to give a constant dosage throughout the extent of the experiment. The second method was to start with a very severe dosage. This is continued for several days, daily weights of the animals being taken. The physical condition of the animal as determined by daily examination was used as an index as to whether it could continue to tolerate a heavy dosage or whether the dosage ought to be gradually diminished. It was determined that an animal is not seriously injured by the loss of several grams in body weight during the first part of an experiment. This second method was devised since it was realized from the study of the results obtained by the first method that the maximum effective dose was greater than the animal would voluntarily take. Even in the first method an increasing amount of fresh water was given to the experimental animals toward the end of the experiment.

The average percentage composition of the mixture used in the experiments listed in this paper is as follows:

Potassium alum 75 gr., Distilled water 400 cc., Concn. HCl 55 cc.

From seven to twelve drops of this solution was put in a small quantity of fresh drinking water each day. Lettuce was included in an otherwise dry diet.

TABLE 1

Table showing the results obtained by the administration of a constant dosage of a mixture of potassium alum and hydrochloric acid by mouth, upon the growth of a transplanted adeno-carcinoma. *dBr* refers to an inbred strain of mice, all individuals of which are uniformly susceptible to the transplant. F_2 refers to the second filial generation produced by originally crossing members of this susceptible inbred strain (*dBr*) to mice from another stock, all members of which are uniformly resistant to the same transplant.

EXPER.	STOCK	CLASS	NUMBER OF MICE	% +	TIME OF STARTING TREATMENT AFTER INOCULATION	TOTAL AVERAGE SURVIVAL TIME AFTER INOCULATION	NOTES	WEIGHT OF TUMOR AT DEATH OF MOUSE	WEIGHT OF MOUSE AT DEATH (TUMOR REMOVED)
1	F_2	Control	1	100		97 days		12.0	21.1
		Exper.	1	100	50 days	201 days	Death accidental	13.8	21.2
2	F_2	Control	1	100		120 days		21.5	24.0
		Exper.	1	100	50 days	120 days	Death (killed) mites	7.9	20.2
3	F_2	Control	6	100		105 days		13.9	22.3
		Exper.	8	100	43 days	155 + days	Two still living	8.7	19.3
4	*dBr*	Control	28	100		74 days		10.7	19.4
		Exper.	28	85.71	10 days	94.7 + days	12 still living	9.3	15.9
5	*dBr*	Control	24	100		44.8		7.8	19.2
		Exper.	32	100	8 days	75.1 + days	8 still living	5.7	15.5

The preliminary work has been done on a slow growing transplantable adeno-carcinoma. This tumor was selected for the initial work on account of its slowness of growth. Any slight deviation from the normal growth rate can be detected since the time element involved is relatively large.

Several difficulties have been encountered in addition to that of getting the animals to take the material. Pneumonia has been very prevalent and has killed off a good many animals. Some have also become infested with mites in spite of periodical dusting with a nicotine preparation. Due to the scratching of the animals over the site of the tumor, some masses have become infected with bacteria. These difficulties have apparently affected both the controls and the treated animals in a like manner.

Results.—The results obtained with the transplantable adeno-carcinoma are furnished in table 1. Since these experiments were preliminary in nature, there has been a considerable variation in the dosage and percentage of the two chemicals employed. For sake of clearness in presentation isolated experiments are grouped together as much as possible. The most important column in the table is that which shows the average survival time (column 7); the lengthening of the life of the mouse by the treatment is very marked.

General Discussion.—Tumors in the treated animals grow at a much slower rate than they do in the controls. This result doubtless may be due in part to a lessened water intake. Since the animals under experimental conditions are able to survive for several weeks longer than the controls, it is reasonable to suppose that this result has been brought about somewhat by the effect of the chemical either upon the host itself or upon the growth of the tumor tissue directly. Just how this slowing up of growth of the transplant is brought about is the subject of further experimentation.

There appears to be a wide range of variation in the ultimate size of the transplants and in the time of death of the animals under treatment. This is yet to be explained. Whether certain animals do not take as much of the mixture as other animals do, or whether there is a variation in the tolerance to the chemical possessed by different animals, or whether both of these factors have an effect has not at the present been determined.

Conclusions.—A mixture of potassium alum and hydrochloric acid administered by mouth so affects the host or the transplant as to bring about a significantly slower growth of the transplant. At the same time, the host is apparently not permanently injured but is able to live longer than it otherwise would if not subjected to treatment.[1]

[1] All the available mice with spontaneous neoplasms have been subjected to treatment with the alum-hydrochloric solution. By the use of the first method of administering the material in the drinking water, it was determined that there was a slowing up of the growth of the tumors, as was the case with the transplantable tumor, although

this point is difficult to determine since direct comparison with a series of control animals is always more or less of an arbitrary procedure. By the second method of administering the material in the drinking water, we have obtained a considerable number of cases of complete regression—a phenomenon we have never witnessed in the controls. We have also brought about the complete regression of the spontaneous mass by the injection of the solution in a greatly reduced concentration subcutaneously. By the combination of administering the material by mouth and by injection we have also witnessed a high percentage of complete regression of the spontaneous tumors. In fact, we have never had a mouse with a spontaneous tumor that did not respond to some extent at least to this double treatment. In all cases, if the treatment be not too severe, the animals are capable of living for several months free of any evidence of new growth after the treatment has been stopped. No tumors have been found in any part of the body on the post mortem examination of such animals. The details of these experiments will be published in the near future.

DYNAMICAL ECONOMICS

By C. F. Roos*

Department of Mathematics, University of Chicago

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1. *Introduction.*—As far back as 1838, Augustin Cournot realized the need for a dynamical theory of economics but chose to avoid the mathematical difficulties by considering a so-called average price and average function of demand.[1] On page 104 of Irving Fisher's doctoral dissertation we find the following statement: "The dynamical side of economics has never yet received systematic treatment. When it has it will reconcile much of the apparent contradiction."[2] In 1901 we find Pareto introducing derivatives of price with respect to time in his analysis.[3] The first American economist to clearly recognize the limitations of the static theory of Cournot and Walras and to suggest a method of overcoming them was H. L. Moore.[4] It was he who first applied the method of relative changes and the method of trend ratios to eliminate long time changes from the law of demand

More recently Irving Fisher has shown that there is a correlation of 0.79 between the volume of trade and a curve representing rapidity of rise or fall of the price level with minor irregularities smoothed out.[5] G. C. Evans has suggested that the demand for a commodity is a function of the price and the rate of change of price.[6]

When the rate of change of price is thus introduced into the demand function the problems of competition and monopoly become problems in the calculus of variations.[7] The monopoly problem is of the classic Lagrange type in the calculus of variations. The competition problem, how-

ever, is a problem dealing with partial maxima of integrals and can be said to be a generalization of the Lagrange problem.[8]

In this paper it is planned to discuss the phenomena of competition, monopoly and coöperation for general functions of demand and cost. We will make use of the following postulates of competition:

1. Each individual engaged in trade is motivated by an attempt to make his net profits as large as possible over a given interval of time.

2. The rate of production of producers 1, 2, ..., $k-1$, $k+1$, ..., n can have no influence on the rate of production of the kth producer except in so much as they cause a change in price which will influence the rate of production of k through the demand function.

2. *Equations of Demand.*—Let us suppose there are n producers manufacturing subject to cost functions $\varphi_1, \varphi_2, \ldots, \varphi_n$ amounts $u_1, u_2, \ldots, u_n$, respectively, of a like commodity in unit time and further that each article sells at a theoretical price p which at any time t is the same for each article produced. Let us suppose further that as many articles are consumed as are produced. As G. C. Evans has suggested the demand Σu_i is often a function of the rate of change of price, dp/dt, as well as of the price. It is conceivable that the demand might also depend upon the rate of production and upon the acceleration of production. In order to insure generality let us write

$$\sum_1^n u_i = g(u_1, \ldots u_n, u_1', u_2', \ldots u_n', p, p', t)$$

where primes denote derivatives with respect to the time. There is then a differential equation

$$\sum_1^n u_i - g(u_1, \ldots u_n, u_1', \ldots, u_n', p, p', t) = 0$$

which we shall call $G(u_1, u_1', \ldots, u_n, u_n', p, p', t) = 0$ relating the price of the commodity and the rates of production of the n producers.

3. *Equations of Cost.*—Inasmuch as the price and rates of production have been chosen as functions of the time the total net profit π_k of each producer for the period of time t_0 to t_1 is given by the integral

$$\pi_k = \int_{t_0}^{t_1} (pu_k - \varphi_k)dt \qquad (k = 1, 2, \ldots, n)$$

where φ_k is a function representing the total cost for the kth producer of manufacturing and distributing per unit time. In general this function φ_k will be a function of the rate of production, the price and the acceleration of production. In order to insure generality let us write

$$\varphi_k = \varphi_k(u_1, u_1', \ldots, u_n, u_n', p, p', t).$$

The integrand of π_k then becomes $pu_k - \varphi_k(u_1, u_1', \ldots, u_n, u_n', p, p', t)$ which we will call $F_k(u_1, u_1', \ldots, u_n, u_n', p, p', t)$ in order to save notation.

4. *Statement of the Problem of Competition.*—The mathematical interpretation of postulate 2 is to regard $u_1, \ldots, u_n$ as independent variables defining p by the equation of demand. A brief statement of the problem is as follows:

Given the end times t_0 and t_1 and the end values of the prices and rates of production fixed at t_0, choose the theoretical price ρ and the rates of production $u_1, \ldots, u_n$ defining a curve Γ in the space $u_1, \ldots, u_n, p, t$ satisfying an equation of demand of the type $G(u_1, u_1', u_2, u_2', \ldots, u_n, u_n', p, p', t) = 0$, such that the profit

$$\pi_k = \int_{t_0}^{t_1} F_k(u_1, u_1', \ldots, u_n, u_n', p, p', t)dt \qquad (k = 1, \ldots, n)$$

of each producer is a maximum when the rate of production u_k alone is allowed to vary with ρ.

It is important to notice that there are two problems involved in the above ambiguous statement. If we know in advance the rates of production at the end of the interval of time $t_0 \leq t \leq t_1$ the problem is a generalized Lagrange problem with fixed end-points. If we wish to analyze data which is presented to us in an attempt to determine a suitable form of the law of demand it is this problem with fixed end-points which is of chief interest, for we know in advance the end values of the quantities $u_1, \ldots, u_n, p, t$.

If, however, we wish to forecast probable rates of production and prices the values of the quantities $u_1, \ldots, u_n, p$ at t_1 are not known in advance and the problem is a generalized Lagrange problem with one end-point available.

5. *First Necessary Conditions.*—To obtain a solution of either of these problems it is convenient to introduce the mathematical concept of partial variations as I have done in another paper.[9] If an n-parameter family of admissible arcs $u_i = u_i(t, a_1, \ldots, a_n)$, $i = 1, \ldots, n$ containing a particular admissible arc Γ for the parametric values $a_1 = a_2 = \ldots = a_n = 0$ be given, the quantities

$$\psi_i(x) = \partial u_i(t, 0, \ldots 0)/\partial a_i$$

are said to be the partial variations of the family along F.

If we assume $\partial G/\partial p' \neq 0$ the partial variations of p are determined by $G = 0$ except for an arbitrary constant as continuous functions of $(t, a_1, \ldots, a_n)$ with continuous derivatives of the first order except possibly at corners. These partial variations satisfy the following partial equations of variations

$$\frac{\partial G}{\partial u_i}\psi_i(x) + \frac{\partial G}{\partial u_i'}\psi_i'(x) + \frac{\partial G}{\partial p}\frac{\partial p}{\partial a_i} + \frac{\partial G}{\partial p'}\frac{\partial p'}{\partial a_i} = 0, \quad i = 1, 2, \ldots, n. \quad (1)$$

The first of these determines $\partial p/\partial a_1$ in terms of ψ_1 and the partial derivatives of G with respect to u_1 and u_1', whereas the second determines $\partial p/\partial a_2$ in terms of ψ_2 and the partial derivatives of G with respect to u_2 and u_2' and so on. Choosing each of the partial variations $\partial p/\partial a_1$, $\partial p/\partial a_2$, . . ., $\partial p/\partial a_n$ equal to zero at $t = t_0$ implies that the total variation of p be zero at t_0, i.e., $\delta p = \partial p/\partial a_1 \delta a_1 + \partial p/\partial a_2 \delta a_2 + \ldots + \partial p/\partial a_n \delta a_n = 0$ at $t = t_0$. The equations of partial variations and the initial conditions, therefore, completely determine the variations of p.

The partial variation of the integral π_k with respect to a_k reduces for $a_1 = a_2 = \ldots a_n = 0$ to the expression

$$\partial\pi_k/\partial a_k \delta a_k = \int_{t_0}^{t_1} [(\partial F_k/\partial u_k)\psi_k + (\partial F_k/\partial u_k)\psi_k' + (\partial F_k/\partial p)(\partial p/\partial a_k) + (\partial F_k/\partial p)(\partial p'/\partial a_k)]\delta a_k dt.$$

By solving the kth differential equation of (1) for $\partial p/\partial a_k$, a first necessary condition analogous to the Euler-Lagrange condition may be obtained without the use of multipliers. Before performing the customary integration by parts it is convenient to apply Dirichlet's formula for changing the order of integration to the iterated integral resulting from the substitution of the value of $\partial p/\partial a_k$ in the equation defining the partial variation of π_k with respect to a_k. It follows readily that the solutions must satisfy the n equations

$$\frac{\partial F_k}{\partial u_k} + \frac{\partial F_k}{\partial p'}\frac{\partial G_p'}{\partial u_k} + \frac{\partial G_p'}{\partial u_k} W_k - \frac{d}{dt}\left(\frac{\partial F_k}{\partial u_k'} + \frac{\partial F_k}{\partial p'}\frac{\partial G_p'}{\partial u_k'} + \frac{\partial G_p'}{\partial u_k'} W_k\right) = 0, \quad (k = 1, \ldots, n) \quad (2)$$

where

$$W_k = \int_{t_0}^{t} e^{\int_x^t \partial G_p'/\partial p\, ds}\left[\frac{\partial F_k}{\partial p} + \frac{\partial F_k}{\partial p'}\frac{\partial G_p'}{\partial p}\right]dx$$

and

$$-\frac{\partial G_p'}{\partial p} = \frac{\partial G/\partial p}{\partial G/\partial p'}; \qquad -\frac{\partial G_p'}{\partial u_k} = \frac{\partial G/\partial u_k}{\partial G/\partial p'}; \qquad -\frac{\partial G_p'}{\partial u_k'} = \frac{\partial G/\partial u_k'}{\partial G/\partial p'}.$$

The extremals of the problem with fixed end-points must, in addition, satisfy the initial conditions $p(t_0) = p_0$, $u_1(t_0) = u_{10}, \ldots, u_n(t_0) = u_{n0}$ $u_1(t_1) = u_{11}, \ldots, u_n(t_1) = u_{n1}$.

The extremals for the problem with one end-point variable must satisfy the initial conditions $p(t_0) = p_0$, $u_1(t_0) = u_{10}, \ldots, u_n(t_0) = u_{n0}$ and the transversality conditions

$$\left[F_k dt + \left(\frac{\partial F_k}{\partial u_k'} + \frac{\partial F_k}{\partial p'}\frac{\partial G_p'}{\partial u_k'} + \frac{\partial G_p'}{\partial u_k'} W_k\right)(du_k - u_k' dt)\right]^{t_1} = 0, \quad (k = 1, \ldots, n). \quad (3)$$

It will be noticed that equations (2) are functional equations for the determination of $u_1, u_2, \ldots, u_n$ and p in terms of the time. The problem is possible inasmuch as we have given initial conditions and the n functional equations plus the one differential equation of demand to determine the $n + 1$ functions $u_1, u_2, \ldots, u_n$ and p.

6. *Further Generalizations.*—Obviously, the first order differential equation suggested in paragraph 2 is not sufficiently general to be taken as the typical relation existing between price, rate of production and time, for very often the demand depends upon the history of prices as well as upon the present price and the rate of change of price. As Karl Karsten points out "purchasers everywhere watch not only the present level of prices, but also the direction in which these prices are changing and are expected to change." He uses the mean between a curve and its first integral as typical of the relation existing between price and demand.[10] I treated the matter in a much more general manner when I wrote the demand as a Volterra integral equation

$$\sum u_i = ap(t) + b + \int_{-\infty}^{t} \varphi(t-\tau)p(\tau)d\tau$$

where $p(-\infty)$ is assumed finite and $\varphi(t-\tau)$ is negligible if $t-\tau$ is large and negative. Such a form of the equation of demand assumes that the demand depends not only on the present price but upon all previous prices as well.[8] As a first approximation we might write $\varphi(t-\tau) = \lambda e^{\alpha(t-\tau)}$ where λ and α are constants.

In the present paper we shall write the demand equation in the even more general form.

$$G(u_1, u_1', \ldots, u_n, u_n', p, p', t) = \int_{t_0}^{t} P(u_1, u_1', \ldots, u_n, u_n', p, p', t, \tau)d\tau \tag{4}$$

where G and P are assumed to be continuous with continuous derivatives of the second order with respect to all of their arguments.

With such a general function of demand the dynamic problem of competition can be stated as follows:

Given the end time t_0 and the end values of the price and rates of production fixed at t_0, choose the price p and the rates of production $u_1, u_2, \ldots, u_n$ defining a curve Γ in the space $u_1, \ldots, u_n, p, t$ satisfying an equation of demand of the type

$$G(u_1, u_1', \ldots, u_n, u_n', p, p', t) = \int_{t_0}^{t} P(u_1, u_1', \ldots, u_n, u_n', p, p, t, \tau)d\tau$$

such that the profit

$$\pi_k = \int_{t_0}^{t_k} F_k(u_1, u_1', \ldots, u_n, u_n', p, p', t)dt$$

of each producer is a maximum when the rate of production u_k alone is allowed to vary with p.

Again there are two problems of interest involved in the above statement of the problem. If we are analyzing data at hand the problem is one with fixed end-points. If we wish to forecast probable rates of production and prices one end-point is variable.

Let us for brevity consider the problem with fixed end-points. Let Γ be the curve defined by $\bar{u}_i(x)$ and $\bar{p}(x)$ if such curves exist and write $u_i(t) = \bar{u}_i(t) + \theta_i(t, a_1, \ldots, a_n)$; $p(t) = \bar{p}(t) + \xi(t, a_1, \ldots, a_n)$ where θ_i and ξ are functions, continuous with their second derivatives with respect to $u_1, u_1', \ldots, u_n, u_n', p, t$ and vanishing when $a_1 = a_2 = \ldots a_n = 0$. If these functions be substituted in the equation (4) it becomes a function of these parameters and yields by a partial differentiation with respect to a_i the equations of partial variation

$$\frac{\partial G}{\partial u_i}\delta\theta_i + \frac{\partial G}{\partial u_i}\delta\theta_i' + \frac{\partial G}{\partial u}\delta\xi + \frac{\partial G}{\partial p'}\delta\xi' = \int_{t_0}^{t}\left[\frac{\partial P}{\partial u_i}\delta\theta_i + \frac{\partial P}{\partial u_i}\delta\theta_i' + \frac{\partial P}{\partial p}\delta\xi + \frac{\partial P}{\partial p'}\delta\xi'\right]d\tau.$$

If we integrate this expression with respect to t and then perform an integration by parts on the primed variations we obtain an expression which by the help of Dirichlet's formula can be written as

$$\frac{\partial G}{\partial p'}\delta\xi = -\frac{\partial G}{\partial u_i}\delta\theta_i + \int_{t_0}^{t}\left[\frac{\partial G}{\partial u_i} - \frac{d}{ds}\frac{\partial G}{\partial u_i'} - \frac{\partial P}{\partial u_i} - \int_s^t\left(\frac{\partial P}{\partial u_i} - \frac{d}{dt}\frac{\partial P}{\partial u_i'}\right)dt\right]\delta\theta_i ds$$

$$\int_{t_0}^{t}\left[\frac{\partial G}{\partial p} - \frac{d}{ds}\frac{\partial G}{\partial p'} - \frac{\partial P}{\partial p'} - \int_s^t\left(\frac{\partial P}{\partial p'} - \frac{d}{dt}\frac{\partial P}{\partial p'}\right)dt\right]\delta\xi ds$$

which is a Volterra integral equation for the determination of $\delta\xi$ in terms of $\delta\theta_i$.

* NATIONAL RESEARCH FELLOW IN MATHEMATICS.

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ON THE PROOF OF SHEPPARD'S CORRECTIONS

BY EDWIN B. WILSON

HARVARD SCHOOL OF PUBLIC HEALTH, BOSTON

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In Whittaker and Robinson's masterly "Calculus of Observations," Art. 99, is a proof of Sheppard's corrections. The crude moments are, by definition,

$$m'_p = \sum_{s=-\infty}^{\infty} x_s^p u_s = \sum_{s=-\infty}^{\infty} x_s^p w \int_{-1/2}^{1/2} f(x_s + nw)\,dn, \qquad (1)$$

where x_s are equidistant abscissas spaced w apart. The frequency function $f(x)$ is treated as extending from $-\infty$ to $+\infty$, as may always be arranged by filling out the undefined range, if any, of $f(x)$ with the definition $f(x) = 0$. The true moment is

$$m_p = \int_{-\infty}^{\infty} x^p f(x)\,dx = w \sum_{s=-\infty}^{\infty} x_s^p f(x_s). \qquad (2)$$

The only assumption stated is that the frequency curve has close contact with the axis at both ends "so that" (2) holds. The left-hand side of (2) is a definition; the right-hand side of (2) is untrue, so far as I can see, for any and every frequency function except for special positions of x_s which cannot be determined in advance.

The proof given by the authors depends on finite differences, and their symbolic manipulation, much in the way in which Sheppard's original proof did (*Proc. London Math. Soc.*, **29**, 1898, 353–380). There are, however, slight differences in the proof and even in the statement of the theorem; for Sheppard assumed a finite range from x_0 to x_p for the frequency function, divided that range into equal intervals and gave his formulas, as usually quoted, merely as approximations of others from which certain terms had been dropped because usually negligible. The terms which are dropped involve, either through summation or differencing, the behavior of the frequency function throughout its whole range. As a practical matter we do not know in advance exactly where the ends of the fitted frequency distribution are. It seems, therefore, worthwhile to reëxamine the whole matter.

If we may assume (2), a derivation of the Sheppard corrections may be given by the simplest of elementary calculus. Interchange the summation and integration in (1); write x_s^p as $(x_s + nw - nw)^p$; expand by the binomial theorem; apply (2); and perform the integration with respect to n.

$$m'_p = \sum_{s=-\infty}^{\infty} x_s^p w \int_{-1/2}^{1/2} f(x_s + nw)\,dn = \int_{-1/2}^{1/2} \sum_{-\infty}^{\infty} w x_s^p f(x_s+nw)\,dn$$

$$= \int_{-1/2}^{1/2} \sum_{-\infty}^{\infty} w(x_s + nw - nw)^p f(x_s + nw)\,dn$$

$$= \int_{-1/2}^{1/2} \sum_{-\infty}^{\infty} w \sum_{i=0}^{p} \frac{(-1)^i p!}{(p-i)!i!}(x_s + nw)^{p-i}(nw)^i f(x_s + nw)\,dn$$

$$= \int_{-1/2}^{1/2} \sum_{0}^{p} \frac{(-1)^i p!}{(p-i)!i!}(nw)^i dn \int_{-\infty}^{\infty} (x_s + nw)^{p-i} f(x_s + nw)dx_s$$

$$= \sum_{0}^{p} \frac{p! w^i m_{p-i}}{(p-i)!(i+1)!2^i}, \qquad i = 0, 2, 4, \ldots \frac{p-1}{2} \text{ or } \frac{p}{2}.$$

When these equations are solved for m in terms of m' the standard form of the Sheppard corrections are found. As $f(x)$ is necessarily positive or zero all that seems to be necessary is that the moments exist, i.e., that $x^p f(x)$ be integrable from $-\infty$ to $+\infty$.

As a matter of fact $w \sum_{-\infty}^{\infty} x_s^p f(x_s)$ is clearly not independent of the size of w nor of the positions of the abscissas x_s, but is a function of x_s with period w. Let

$$\psi_p(x_s) = w \sum_{-\infty}^{\infty} x_s^p f(x_s), \quad A_{k,p} = \frac{2}{w} \int_{-w/2}^{w/2} w \sum_{-\infty}^{\infty} x_s^p f(x_s) \cos \frac{2\pi k x_s}{w} dx_s$$

with a similar expression $B_{k,p}$ in the sine. Each integration extends over an interval w centered at x_s and the summation adds these results from $-\infty$ to $+\infty$. Hence

$$A_{k,p} = 2 \int_{-\infty}^{\infty} x_s^p f(x_s) \cos \frac{2\pi k x_s}{w} dx_s, \qquad B_{k,p} = 2 \int_{-\infty}^{\infty} x_s^p f(x_s) \sin \frac{2\pi k x_s}{w} dx_s$$

$$\psi_p(x_s) = \frac{1}{2} A_0 + \sum_{1}^{\infty} A_{k,p} \cos 2\pi k x_s/w + \sum_{1}^{\infty} B_{k,p} \sin 2\pi k x_s/w.$$

Or

$$w \sum_{-\infty}^{\infty} x_s^p f(x_s) = m_p + \sum_{1}^{\infty} A_{k,p} \cos 2\pi k x_s/w + \sum_{1}^{\infty} B_{k,p} \sin 2\pi k x_s/w. \quad (3)$$

This equation takes the place of (2) and is exact. The function $x^p f(x)$ does not need to be continuous, it does need to satisfy some slight condition of developability into Fourier's series, such as being of limited variation, which would always be fulfilled by any function we should use as a frequency function.

To illustrate, take the normal frequency function. Here

$$\frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{-x^2/2\sigma^2} \cos \frac{2\pi k x}{w} dx = e^{-2\pi^2 k^2 \sigma^2/w^2} = \frac{1}{2} A_{k,0}, \qquad B_{k,0} = 0$$

$$\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} xe^{-x^2/2\sigma^2}\sin\frac{2\pi kx}{w}\,dx = \frac{2\pi k\sigma^2}{w}\,e^{-2\pi^2k^2\sigma^2/w^2} = \frac{1}{2}B_{k,1}, \qquad A_{k,1} = 0$$

$$w\sum_{-\infty}^{\infty} f(x_s) = 1 + 2\sum_{1}^{\infty} e^{-2\pi^2k^2\sigma^2/w^2}\cos\frac{2\pi kx_s}{w} = \Theta_1(x_s), \qquad q = e^{-2\pi^2\sigma^2/w^2}$$

$$w\Sigma x_s f(x_s) = 0 + 2\sum_{1}^{\infty}\frac{2\pi k\sigma^2}{w}\,e^{-2\pi^2k^2\sigma^2/w^2}\sin\frac{2\pi kx_s}{w} = -\sigma^2\Theta_1'(x_s),$$

where Θ_1 is the Jacobi function (see, e.g., Wilson, "Advanced Calculus," p. 469). If q is small the Θ-functions converge with extreme rapidity. Even if $w = \pi\sigma$, which is much larger than would ever be taken in practice, $q = e^{-2} = 0.13533$ and the series may be reduced to the term in q, neglecting $q^4, \ldots$. The even moments m' may be obtained by applying repeatedly the operator

$$-\left(\frac{w}{2\pi}D_k\right)^2 \text{ to } e^{-2\pi^2k^2\sigma^2/w^2}, \qquad \frac{1}{2}A_{k,2i} = (-1)^i\left(\frac{w}{2\pi}\frac{d}{dk}\right)^{2i} e^{-\pi^2k^2\sigma^2/w^2}$$

$$m_2' = \sigma^2\left[1 + 2\sum_{1}^{\infty} e^{-2\pi^2k^2\sigma^2/w^2}\left(1 - \frac{4\pi^2k^2\sigma^2}{w^2}\right)\cos\frac{2\pi kx_s}{w}\right] = \sigma^2\,[\Theta_1(x_s) + \sigma^2\Theta_1''(x_s)].$$

It may be observed that the higher we go up in the series of moments, the more important does the correction become, the multipliers of the trigonometric series being of the order $(2\pi k\sigma/w)^p$ for the pth moment.

To go back and correct the proof of Sheppard's corrections we have merely to correct the last steps by inserting the proper value (3) of the infinite sum instead of (2). Then

$$m_p' = \sum_{0}^{p}\frac{(-1)^i p!w^i}{(p-i)!i!}\int_{-1/2}^{1/2} n^i dn\left[m_{p-i} + \sum_{k=1}^{\infty} A_{k,p-i}\cos\frac{2\pi k(x_s+nw)}{w} + \sum_{k=1}^{\infty} B_{k,p-i}\sin\frac{2\pi k(x_s+nw)}{w}\right].$$

The trigonometric terms may be expanded. Let

$$\int_{-1/2}^{1/2} n^i\cos 2\pi kn\,dn = I_{k,i}, \qquad \int_{-1/2}^{1/2} n^i\sin 2\pi kn\,dn = J_{k,i},$$

where for i odd we have $I_{k,i} = 0$ and for i even we have $J_{k,i} = 0$. Then

$$m_p' = \sum_{0}^{p}\frac{p!w^i m_{p-i}}{(p-i)!(i+1)!2^i} \qquad i \text{ even}$$

$$+ \sum_{2}^{p}\frac{p!w^i}{(p-i)!i!}\sum_{k=1}^{\infty} I_{k,i}\left(A_{k,p-i}\cos\frac{2\pi kx_s}{w} + B_{k,p-i}\sin\frac{2\pi kx_s}{w}\right) \qquad i \text{ even}$$

$$- \sum_{1}^{p}\frac{p!w^i}{(p-i)!i!}\sum_{k=1}^{\infty} J_{k,i}\left(B_{k,p-i}\cos\frac{2\pi kx_s}{w} - A_{k,p-i}\sin\frac{2\pi kx_s}{w}\right), \qquad i \text{ odd.}$$

Where, as indicated, i runs on even integers from 0 to $(p-1)/2$ or $p/2$ as the case may be, in the first line, and on even integers from 2 up in the second, but on odd integers from 1 up in the third.

The last two lines give the corrections for Sheppard's corrections due to the inexactness of (2) and take the place of Sheppard's own form of them in terms of differences. It is particularly to be observed that the mean value of the trigonometric correction terms is zero, i.e., the usual formulas for Sheppard's corrections are true on the average for any frequency function, for which the highest moments considered exist, without any consideration of how the function behaves at the ends of its range or within it. The only reason, then, for considering the behavior at the ends must be of a practical, rather than theoretical nature connected with the possibility that, if the contacts at the ends are not close, the trigonometric correction terms are of the same order of magnitude as the Sheppard corrections, so that in the absence of any knowledge of the former, it is useless to apply the latter in any particular example despite the fact that in the average the former vanish.

To illustrate, consider the function $f(x) = 1$, $-1/2 < x < 1/2$, $f(x) = 0$, $k > 1/2$ or $x < -1/2$. This has no contact at either end of its true range, though if we include the artificial range the end contact is perfect but the intermediate behavior discontinuous. It need not be assumed that we know in advance the positions of the ends. We could have either of the following two histograms from grouped observations:

I	x	$-2/3$ to $-1/3$	$-1/3$ to 0	0 to $1/3$	$1/3$ to $2/3$
	F	0.167	0.333	0.333	0.167
II	x	$-1/2$ to $-1/6$	$-1/6$ to $1/6$	$1/6$ to $1/2$	
	F	0.333	0.333	0.333	

and similarly for other values of w than $1/2$. The coefficients are

$$A_{k,0} = 2\int_{-1/2}^{1/2} \cos\frac{2k\pi x}{w}\,dx = \frac{2w}{k\pi}\sin\frac{k\pi}{w}, \qquad B_{k,0} = 0,$$

$$B_{k,1} = 2\int_{-1/2}^{1/2} x\,\sin\frac{2k\pi x}{w}\,dx = \left(\frac{w}{k\pi}\right)^2 \sin\frac{k\pi}{w} - \frac{w}{k\pi}\cos\frac{k\pi}{w}, \qquad A_{k,1} = 0.$$

The values of $I_{k,i}$ and $J_{k,i}$ are always as follows:

$$I_{k,i} = \frac{2i(-1)^k}{(2\pi k)^{i+1}}\left[(\pi k)^{i-1} - (i-1)(i-2)(\pi k)^{i-3} + (i-1)(i-2)(i-3)(i-4)(\pi k)^{i-5}\ldots\right], \quad i \text{ even}$$

$$J_{k,i} = \frac{-2(-1)^k}{(2\pi k)^{i+1}}\left[(\pi k)^i - i(i-1)(\pi k)^{i-2} + i(i-1)(i-2)(i-3)(\pi k)^{i-4}\ldots\right], \quad i \text{ odd}$$

Then

$$m_1' = 0 - w\sum_1^\infty \frac{(-1)^k}{2\pi k}\,\frac{2w}{\pi k}\,\sin\frac{k\pi}{w}\sin\frac{2\pi k x_s}{w}.$$

The correction is quite 0 if $1/w$ is an integer. The series is

$$m_1' = \frac{w^2}{\pi^2}\left[\sin\frac{\pi}{w}\sin\frac{2\pi k x_s}{w} - \frac{1}{4}\sin\frac{2\pi}{w}\sin\frac{4\pi k x_s}{w} + \frac{1}{9}\sin\frac{3\pi}{w}\sin\frac{6\pi k x_s}{w}\ldots\right].$$

It is seen that the periodic error in m_1' is of the order of magnitude of w^2/π^2 at most. As w would always be small compared to 1, the correction would be very small. In respect to the second moment

$$m_2' = m_2 + \frac{w^2}{12} + w^2 \sum_{k=1}^{\infty} \frac{(-1)^k}{2(\pi k)^2} \frac{2w}{\pi k} \sin \frac{k\pi}{w} \cos \frac{2\pi k x_s}{w}$$

$$+ 2w \sum_{k=1}^{\infty} \frac{(-1)^k}{2\pi k} \left(\frac{w^2}{(\pi k)^2} \sin \frac{k\pi}{w} - \frac{w}{\pi k} \cos \frac{k\pi}{w} \right) \cos \frac{2\pi k x_s}{w},$$

or

$$m_2' = m_2 + \frac{w^2}{12} - \frac{w^2}{\pi^2} \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} \cos \frac{k\pi}{w} \cos \frac{2\pi k x_s}{w}$$

$$+ \frac{2w^3}{\pi^3} \sum_{k=1}^{\infty} \frac{(-1)^k}{k^3} \sin \frac{k\pi}{w} \cos \frac{2\pi k x_s}{w}.$$

There is a trigonometric series of the second order in w/π and one of the third order. The second order term, which will vanish, on the average, is, at its greatest, of just about the magnitude of the Sheppard correction $w^2/12$, but may either cancel or double it. To go on to higher moments would show that a second order trigonometric correction occurs in them all.

As an example of a skew distribution take $f(x) = 0$, $x < 0$, $f(x) = e^{-x}$, $x \geqq 0$. This type is approached by many extremely skew frequency functions which drop off sharply on the left hand. With $w = 1$ we might have for grouped data, if the starting points or origin were unknown in advance, either of these histograms.

I	x	0–1	1–2	2–3	3–4	4–5	5–6	6–7
	F	0.632	0.233	0.086	0.031	0.012	0.004	0.001
II	x	$-1/2$ to $1/2$	$1/2$ to $3/2$	$3/2$ to $5/2$	$5/2$ to $7/2$	$7/2$ to $9/2$	$9/2$ to $11/2$	$11/2$ to $13/2$
	F	0.393	0.383	0.143	0.050	0.019	0.007	0.003

The series for m_1' and m_2' are readily summed exactly in these cases and the errors from $m_1 = 1.00$, $m_2 = 100$ may be determined. The general analytical calculation gives the following results:

$$A_{k,0} = 2\int_0^{\infty} e^{-x} \cos \frac{2k\pi x}{w}\, dx = \frac{2}{1 + (2k\pi/w)^2}, \qquad B_{k,0} = 2\int_0^{\infty} e^{-x} \sin \frac{2k\pi x}{w}\, dx = \frac{4k\pi/w}{1 + (2k\pi/w)^2}$$

$$A_{k,1} = \frac{w}{2\pi} \frac{d}{dk} B_{k,0} = \frac{2 - 2(2k\pi/w)^2}{(1 + (2k\pi/w)^2)^2}, \qquad B_{k,1} = -\frac{w}{2\pi} \frac{d}{dk} A_{k,0} = \frac{8k\pi/w}{1 + (2k\pi/w)^2)^2}.$$

Then

$$m_1' = m_1 + w \sum_{k=1}^{\infty} \frac{(-1)^k}{2\pi k} \left(\frac{4k\pi/w}{1 + (2k\pi/w)^2} \cos \frac{2\pi k x_s}{w} - \frac{2}{1 + (2k\pi/w)^2} \sin \frac{2\pi k x_s}{w} \right).$$

If w is small compared with 2π, this may be written

$$m_1' = m_1 + \frac{w^2}{2\pi^2} \sum_{1}^{\infty} \frac{(-1)^k}{k^2} \cos \frac{2\pi k x_s}{w} - \frac{w^3}{\pi^3} \sum_{1}^{\infty} \frac{(-1)^k}{k^3} \sin \frac{2\pi k x_s}{w}.$$

There are trigonometric terms of the second and third orders in w/π. In the case of the histogram II with $x_s = 0$, $m_1' = m_1 - 0.05$ approximately, and for I with $x_s = 0.5$, $m_1' = m_1 + 0.08$, approximately. The values of $A_{k,1}$ and $B_{k,1}$ are needed for m_2'. Again there are errors of the second order in w/π. The values of the second moment about the mean in the two histograms are 1.09 for I and 0.84 for II with the Sheppard correction $w^2/12$ but 1.17 and 0.92 without. If we do not apply the corrections the value of the

standard deviation varies from 8% too much to 4% too little, without the correction but from 4% too much to 8% too little if we apply it, according to the way we make up the histogram. It is doubtful if in practice w would be taken so large as 1 in such a case as this, but the dilemma would remain the same except on a reduced scale.

The illustrations have been extreme in that they possessed major discontinuities. It is probably safe to apply the Sheppard corrections in all cases of mere doubt about the contact at the ends. Whether it is worthwhile to apply the corrections at all when consideration is had for the sampling errors is another matter. For example, the relative error in σ^2 is $(2/n)^{1/2}$ and if this is to be less than $w^2/12\sigma^2$, the value of n must exceed $288\sigma^4/w^4$. On the other hand if $f(x)$ is practically normal, the expansion for the trigonometric corrections is very rapidly convergent, as is illustrated above, and reasonably large values of w, larger than are ordinarily used in practice, give such good values of the moments that, with the Sheppard corrections, the parameters of the frequency functions may be determined well within their sampling errors, even for n large, with relatively few intervals. For example, Whittaker and Robinson, p. 189, calculate the mean and standard deviation for a set of 10,000 chest measurements given to the nearest inch. If the grouping had been in 3-inch intervals we should have mean = 39.842 in place of 39.835 ± 0.02 and $\sigma = 2.04$ in place of 2.05 ± 0.015.

NOTE ON A GENERALIZATION OF TAYLOR'S SERIES

BY D. V. WIDDER*

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CHICAGO

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1. It is a familiar fact that if one determines the constants $c_0, c_1 \ldots, c_n$ of the polynomial

$$s_n(x) = c_0 + c_1x + c_2x^2 + \ldots + c_nx^n$$

in such a way that the curve $y = s_n(x)$ shall have closest contact with a curve $y = f(x)$ at a point $x = t$, one obtains the first $(n + 1)$ terms of the Taylor development of $f(x)$,

$$s_n(x) = f(t) + f'(t)(x-t) + \ldots + f^{(n)}(t)\ \frac{(x-t)^n}{n!}.$$

The series

$$\sum_{n=0}^{\infty} [s_n(x) - s_{n-1}(x)] \qquad (1)$$

is then Taylor's series. Heres $s_n(x)$ may be regarded as a linear combination of the first $(n + 1)$ terms of the sequence

$$1, x, x^2, \ldots . \tag{2}$$

If one proceeds in a similar way with the sequence

$$1, \sin x, \cos x, \sin 2x, \cos 2x, \ldots . , \tag{3}$$

one is led to a series studied by G. Teixeira[1] in another connection,

$$f(x) \sim \sum_{n=0}^{\infty} A_n[1-\cos x]^n + B_n[1-\cos x]^n \sin x$$

$$A_n = \frac{2^n}{(2n)!} D^2(D^2 + 1^2)(D^2 + 2^2). \ldots .(D^2 + \overline{n-1}^2)f(0), \qquad B_n = D(D^2 + 1^2)(D^2 + 2^2). \ldots .(D^2 + n^2)f(0),$$

where D indicates the operation of differentiation.

It seems natural then to consider the sequences (2) and (3) as special cases of a more general sequence

$$u_0(x), u_1(x), u_2(x), \ldots . , \tag{4}$$

and to investigate the nature of the series introduced by the method of approximation employed above. The purpose of the present note is to give an outline of a general theory that can be developed along these lines, the details of which the author will publish later.

2. The functions (4) are considered to be real functions of the real variable defined in a common interval $a \leqq x \leqq b$.

Definition.—The function

$$s_n(x) = \sum_{i=0}^{n} c_i u_i(x)$$

is a function of approximation of order n for the point $x = t$ if the functions $u_i(x)$ are of class C^n in the neighborhood of $x = t$, and if $s_n(x)$ has contact of order n at least with $f(x)$ at $x = t$.

Denote the Wronskian of the functions $v_0(x), v_1(x), \ldots . , v_n(x)$ by

$$W[v_0(x), v_1(x), \ldots . , v_n(x)] = \left| v_i^{(k)}(x) \right|$$

and in particular for the functions of the set (4) write

$$W_n(x) = W[u_0(x), u_1(x), \ldots . , u_n(x)].$$

THEOREM I. *If the functions $f(x), u_0(x), u_1(x), \ldots . , u_n(x)$ are of class C^n in the neighborhood of $x = t$, and if $W_n(t) \neq 0$, then there exists a unique function of approximation of order n for $x = t$.*

The series (1) now takes the form

$$\sum_{n=0}^{\infty} [s_n(x) - s_{n-1}(x)] = \sum_{n=0}^{\infty} L_n f(t) g_n(x, t), \tag{5}$$

where L_n is a linear differential operator defined by the relation

$$L_n f(x) = \frac{W[u_0(x), u_1(x), \ldots, u_{n-1}(x), f(x)]}{W_{n-1}(x)},$$

and where $g_n(x, t)$ is the function of Cauchy,[2]

$$g_n(x, t) = \frac{1}{W_n(t)} \begin{vmatrix} u_0 & (t) & u_1(t) & \ldots & u_n(t) \\ u_0' & (t) & u_1'(t) & \ldots & u_n'(t) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ u_0^{(n-1)}(t) & & u_1^{(n-1)}(t) & \ldots & u_n^{(n-1)}(t) \\ u_0 & (x) & u_1(x) & \ldots & u_n(x) \end{vmatrix}.$$

Clearly series (5) reduces to Taylor's series for the sequence (2).

THEOREM II. *Let the functions $f(x)$, $u_0(x)$, $u_1(x)$, ..., $u_n(x)$ be of class C^{n+1}, and let the Wronskians $W_0(x)$, $W_1(x)$, ..., $W_n(x)$ be positive in the interval $a \leqq x \leqq b$. Then if t is a point of this interval,*

$$f(x) = \sum_{k=0}^{n} L_k f(t) g_k(x, t) + R_n(x),$$

where $R_n(x)$ has one of the forms

$$R_n(x) = \int_t^x g_n(x, t) L_{n+1} f(t) dt,$$

$$R_n(x) = \frac{g_n(x, \xi)}{g_m(x, \xi)} L_{n+1} f(\xi) \int_t^x g_m(x, s) ds, \qquad m \leqq n, \quad \begin{matrix} t < \xi < x, \\ t > \xi > x. \end{matrix}$$

3. Define a set of functions $\phi_k(x)$ by the equations

$$\phi_0(x) = W_0(x), \qquad \phi_1(x) = \frac{W_1(x)}{[W_0(x)]^2},$$

$$\phi_k(x) = \frac{W_k(x) W_{k-2}(x)}{[W_{k-1}(x)]^2}, \qquad k = 2, 3, \ldots.$$

Now introduce the

Conditions A.—(*a*) The functions $\phi_i(x)$ are of class C^{∞} in the interval $a \leqq x \leqq b$,

(*b*) $$\phi_i(x) > 0, \qquad a \leqq x \leqq b,$$

(*c*) $$\lim \frac{M_n}{m_n} = 1,$$

where

M_n = *maximum* $\phi_n(x)$, m_n = *minimum* $\phi_n(x)$ in $a \leqq x \leqq b$.

THEOREM III. *If the functions $\phi_i(x)$ satisfy Conditions A in (a, b), and if the series*

$$\sum_{n=0}^{\infty} c_n g_n(x, t), \qquad a \leqq t \leqq b,$$

converges for a value $x = x_0 \neq t$ of the interval $a \leqq x \leqq b$, then it converges absolutely in the interval $|x-t| < |x_0-t|$, $a \leqq x \leqq b$ and uniformly in any closed interval included therein. If the sum of the series is denoted by $f(x)$, then

$$L_k f(x) = \sum_{n=0}^{\infty} c_n L_k g_n(x, t), \qquad k = 0, 1, 2, \ldots, |x-t| < |x_0-t|, \qquad a \leqq x \leqq b.$$

THEOREM IV. *If the functions $\phi_i(x)$ satisfy Conditions A, and if*

$$f(x) = \sum_{n=0}^{\infty} c_n g_n(x, u), \qquad |x-u| < r, \qquad a \leqq x \leqq b,$$

then

$$f(x) = \sum_{n=0}^{\infty} L_n f(t) g_n(x, t), \qquad u-r < t < u+r$$

for all x and t satisfying the relation $|x-t| < r-|t-u|$, $a \leqq x \leqq b$.

This theorem evidently generalizes the process of analytic continuation.

THEOREM V. *Let the functions $\phi_i(x)$ satisfy Conditions A in (a, b). Then a necessary and sufficient condition that a function $f(x)$ defined in the interval $a \leqq x \leqq b$ can be represented by a series*

$$f(x) = \sum_{n=0}^{\infty} L_n f(a) g_n(x, a), \qquad a \leqq x < b$$

is that $f(x)$ be the difference of two functions of class C^{∞} in $a \leqq x < b$,

$$f(x) = \phi(x) - \psi(x)$$

such that

$$L_n\phi(x) > 0, \text{ or } \phi(x) \equiv 0; L_n\psi(x) > 0, \text{ or } \psi(x) \equiv 0, \quad a < x < b, \quad n = 0, 1, \ldots$$

This theorem is a generalization of a theorem of S. Bernstein.[3] We now define

Conditions B.—(*a*) Conditions A are satisfied in (a, b),

$$(b) \quad \frac{d^k}{dx^k}\left(\frac{1}{\phi_n(x)}\right) \geqq 0, \qquad k = 1, 2, 3, \ldots, \quad n = 0, 1, 2, \ldots, \quad a \leqq x \leqq b.$$

THEOREM VI. *If conditions B are satisfied in (a, b), and if $f(x)$ is analytic in $a < x < b$, then*

$$f(x) = \sum_{n=0}^{\infty} L_n f(t) g_n(x, t), \qquad a < t < b, \tag{6}$$

the series being convergent in some (two-sided) neighborhood of t.

Finally, to show that the series (6) is a *bona fide* generalization of Taylor's series, note that for $\phi_i(x) = 1$, for which Conditions B are satisfied, (6) reduces to Taylor's series. Conditions B are also satisfied if

$$\phi_i(x) = e^{-\frac{x}{n}},$$

and for these values one is led to a series quite different from Taylor's series.

* NATIONAL RESEARCH FELLOW.

[1] *Bul. Sci. Math. Astr.*, **25**, p. 200.

[2] E. Goursat, *Cours d'Analyse Mathématique*, vol. 2, p. 430.

[3] *Math. Ann.*, **75**, p. 449.

A THEORY OF MATTER AND ELECTRICITY[1]

BY GEORGE D. BIRKHOFF

DEPARTMENT OF MATHEMATICS, HARVARD UNIVERSITY

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Up to the present time no mathematical theory of matter and electricity seems to have been proposed which meets the fundamental demands of *determinateness* and *stability*. A theory which appears to satisfy these demands is presented herewith. In a second following note it is proved that this theory leads to a formula of the Balmer type for the frequencies of the small oscillations of a hydrogen atom.

1. *The Perfect Fluid.*—We consider first the space-time of the special theory of relativity with time coördinate x_1 and rectangular space coördinates x_2, x_3, x_4, the units of length and time being so chosen that the velocity of light is 1. By an "adiabatic fluid" is meant matter whose state is determined by a functionally related pressure and density, p and ρ such that if $u^i = dx_i/ds$ denotes the velocity tensor, and if the "energy tensor" is defined to be

$$T^{ij} = \rho u^i u^j - p g^{ij} \qquad (g^{ij} = 0,\ i \neq j;\ g^{11} = -g^{22} = -g^{33} = -g^{44} = 1), \tag{1}$$

then the equations of motion are obtained by setting

$$\frac{\partial T^{i\alpha}}{\partial x_\alpha} = 0. \tag{2}$$

These equations express the relativistic form of the principles of conservation of energy ($i = 1$) and of linear momentum in the direction of the three axes ($i = 2, 3, 4$). From them it appears that a certain quantity

$$\int \rho e^{-\int \frac{dp}{\rho}} d\tau \quad (d\tau, \text{ element of volume}) \tag{3}$$

has always the same value over any given portion of the fluid;[2] for example, if $p = \rho$, it is the volume $\int d\tau$ which is invariable. Under the action of an arbitrary body force tensor f^i, the only modification necessary is to replace the right-hand member of (2) by f^i. It is understood that the only forces allowed must be orthogonal to the velocity tensor, i.e., $f^\alpha u_\alpha = 0$.

A possible state of equilibrium of such a fluid is that of constant pressure and density. If one computes the velocity of a slight disturbance from this state, it turns out to be $a^{1/2}/(1-a)^{1/2}$ where we write $a = dp/d\rho$.[3] Of course, the velocity of such an elastic disturbance cannot exceed that of light. On the other hand, if that velocity is less than that of light, difficulties of indeterminateness seem to arise when two portions of the adiabatic fluid collide at relative velocities sufficiently near the velocity of light. In order to avoid this difficulty of indeterminateness, it seems to be necessary to have a disturbance velocity equal to that of light at all pressures and densities. Such a fluid, for which $p = {}^1/_2\rho$, will be termed the "perfect fluid," by analogy with an ordinary perfect gas.

2. *Electricity and the Perfect Fluid.*—If F_{ij} is a skew-symmetric tensor of the second order so that $F_{ij} = -F_{ji}$ then the tensor

$$f^i = F^{i\alpha} u_\alpha$$

yields a force tensor which is in all cases orthogonal to the velocity tensor. We will regard F_{12}, F_{13}, F_{14} as the components of electric force and $-F_{34}$ $-F_{42}$, $-F_{23}$ as the components of magnetic force. The well-known Maxwell-Lorentz equations may then be written in the form

$$\frac{\partial F_{ij}}{\partial x_k} + \frac{\partial F_{jk}}{\partial x_i} + \frac{\partial F_{ki}}{\partial x_j} = 0, \quad \frac{\partial F_i{}^\alpha}{\partial x_\alpha} = -4\pi\sigma u_i, \tag{4}$$

where σ is the density of electricity.

From these it follows that the total charge σ is invariable, so that electricity may be regarded as a substance.

Now we suppose that the perfect fluid is permanently charged with electricity of which it acts as the carrier. More precisely if we define the "electromagnetic energy tensor" as

$$E_{ij} = F_i{}^\alpha F_{\alpha j} + \frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} g_{ij}, \tag{5}$$

the ponderomotive equations for combined matter and electricity take a form like (2) except that the energy tensor T_{ij} of matter is replaced by $T_{ij} + E_{ij}$, i.e., by the combined energy tensor of the perfect fluid and of electricity.

Since the quantity (3), as well as the electrical charge is invariable it

follows that these must remain in a constant ratio along any world line of the fluid. For the case of the perfect fluid this leads at once to the equation (6)

$$\rho = \frac{\varphi^2\sigma^2}{\pi}. \tag{6}$$

Here $\varphi/\sqrt{\pi}$ is the constant ratio referred to with the divisor $\sqrt{\pi}$ introduced for the sake of convenience. The quantity φ will be termed the "substance coefficient."[4] It is an arbitrary function of position, specified once for all at the outset and remaining fixed in value along each world line of the fluid.

Since electricity of one sign repels itself, and since the perfect fluid tends to expand under the enormous elastic pressures, the charged perfect fluid tends to expand all the more with velocities which approach that of light. Under such expansion the mass of the relativistic fluid approaches zero since the density of matter varies as the square of the density of electricity by (6).

3. *Atomic Potential Energy.*—As a first step toward relieving the instability just referred to we propose to define an "atomic potential" ψ, given once for all at each particle of the fluid. In this case along the world line of a particle we have

$$\frac{d\psi}{ds} = \frac{\partial\psi}{\partial x_\alpha} u^\alpha = 0$$

so that the tensor $\partial\psi/\partial x_i$ may function as a second force tensor, being orthogonal to the velocity tensor. It is this further body force which we shall assume to be present in the fluid. If we define

$$A_{ij} = \psi g_{ij} \tag{7}$$

as the "atomic potential energy tensor," then the laws of motion for the perfect fluid under elastic, electrical and atomic potential forces may be expressed in the form (2) except that T_{ij} is replaced by the complete energy tensor

$$H_{ij} = T_{ij} + E_{ij} + A_{ij},$$

i.e., the equations of motion are

$$\frac{\partial H^{i\alpha}}{\partial x_\alpha} = 0. \tag{8}$$

It is to be remembered that in addition to these equations there is the constitutive equation $p = {}^1/_2\rho$ as well as the Maxwell-Lorentz equations (4), and the equation

$$\frac{\partial\psi}{\partial x_\alpha} u^\alpha = 0, \tag{9}$$

expressing the fact that ψ is constant along each world line.

It would seem to be desirable that ψ be required to vanish along the free boundary of any portion of the perfect fluid. Otherwise there will be an inward normal pressure equal to ψ, and the resultant of the internal body forces due to the atomic potential will not vanish.

4. *The Conservation of Energy.*—If one transforms properly the equation (8) for $i = 1$, it leads to a principle of the conservation of energy at low velocities in which

$$V = \int \left[\rho - p + \frac{1}{8\pi} (F_{12}^2 + \ldots + F_{23}^2) + \psi \right] d\tau \qquad (10)$$

figures as potential energy, so that only half the mass counts an elastic energy. The electromagnetic energy density has the usual form.

The atomic potential energy is proportional to the volume, and in consequence the volume cannot increase indefinitely. Here then is a first factor operating in the direction of stability. A portion of the fluid of one sign would tend to scatter in wisplike form in space rather than to fill all of space.

For $i = 2, 3, 4$ the equation (8) expresses the principles of the conservation of linear momentum in the direction of the x_2, x_3, x_4 axes, respectively.

5. *Gravitation.*—If now, in accordance with Einstein's general scheme for taking care of gravitational phenomena, we write

$$R_{ij} - \frac{1}{2} R g_{ij} = - 8\pi H_{ij}, \qquad (11)$$

and employ the same laws as before in the sense of the "principle of equivalence," an appropriate extension of the above theory of matter and electricity to the space-time of the general theory of relativity results. As is well known, the relations (8) then appear as implicit in (11). The gravitational effect of matter and energy is, however, not exactly proportional to the energy density (see (10)) but is instead given by

$$\rho + \frac{1}{8\pi} (F_{12}^2 + \ldots + F_{23}^2) + \psi. \qquad (12)$$

6. *The Structure of Matter. Stability.*—To obtain a theory of matter satisfying the fundamental requirements of determinateness and stability we may now proceed as follows.

Let us suppose that the protons are initially portions of the perfect fluid carrying a charge e, and of one or more types of spherical distribution of φ, ψ, σ. These protons may collide with one another or with themselves, but by definition are not to interpenetrate.

Similarly, certain other portions of the perfect fluid of one particular type carrying a charge $-e$ constitute the electrons. Their mass is much less and the volume much greater than that of the protons. These may collide with one another, but by definition they cannot interpenetrate.

If now we were to require that electrons and protons cannot interpenetrate it is readily proved that they tend to wisplike forms of finite volume with nearly neutralizing portions of positive and negative electricity in every part. The natural way in which to avoid this difficulty of an amorphous stable condition is to allow the free interpenetration of electron and proton. It is almost as though the proton were taken as a point charge which freely penetrates the electron, although this limiting case introduces the vital difficulty of infinite available energy.

We, therefore, require the free interpenetration of the proton and electron. However, there can be no three fluids overlapping at a point, since two of the same sign cannot overlap. The appropriate modification in the equations of motion is easily made. In the Maxwell-Lorentz equations (4) the current density tensor σu_i is replaced by $\sigma_+ u_i + \sigma_- v_i$ where σ_+ is the density of positive charge with velocity tensor u_i while σ_- is the density of the negative charge with velocity tensor v_i. Similarly the energy tensors T_{ij} and A_{ij} are modified, respectively, to

$$T_{ij}^* = \rho_+ u_i u_j + \rho_- v_i v_j - (p_+ + p_-)g_{ij}, \; A_{ij}^* = (\psi_+ + \psi_-)g_{ij}$$

where the meaning of the notation is obvious.

The equations (8) express as before the four conservation conditions, and follow at once from the equations of motion for the interpenetrating fluids. For example, for the proton these are

$$\frac{\partial T^{i\alpha}}{\partial x_\alpha} = f^i$$

where f^i is the electromagnetic force tensor for the proton,

$$f_i = \sigma_+ F_{i\alpha} u^\alpha.$$

In this way, even in the space-time of the general theory of relativity, the modified equations are at once obtained.

With such a set of requirements it is not hard to see that a position of stable equilibrium of any system of protons and electrons can be expected to appear. In fact none of the protons or electrons can contract indefinitely since that would require an infinite elastic potential energy. Nor can they expand indefinitely because of the atomic potential energy. And, finally, the electrical attractions between the oppositely charged protons and electrons, and their free interpenetration, tend to make them coalesce into neutral stable forms made up of a number of connected pairs of protons and electrons forming the individual atoms.

[1] An outline of this theory was presented in an address entitled "A Mathematical Critique of Some Physical Theories," given before a meeting of Section A of the American Association for the Advancement of Science, The American Mathematical Society and the Mathematical Association of America on Dec. 30, 1926. The address will appear in full in an early number of the *Bulletin of the American Mathematical Society*.

[2] See my book, *Relativity and Modern Physics*, p. 100.
[3] Pp. 104–105, loc. cit. 2.
[4] See my book, *The Origin, Nature and Influence of Relativity*, chap. 7.

THE HYDROGEN ATOM AND THE BALMER FORMULA

BY GEORGE D. BIRKHOFF

DEPARTMENT OF MATHEMATICS, HARVARD UNIVERSITY

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I propose to apply the theory of the preceding note to the consideration of the small oscillations of the proton and single electron forming the hydrogen atom. On making certain simple assumptions about the substance coefficients and atomic potentials at our disposal, characteristic difference frequencies are obtained which are of the same type as those given by the Balmer formula.[1]

1. *The State of Equilibrium.*—As stated at the end of the preceding note, we may expect more or less complete coalescence of the electron and proton in the spherical condition of equilibrium. Let $\sigma = \sigma_+ + \sigma_-$[2] stand for the total density of charge. The radial electric force is clearly

$$F = \frac{4\pi \int_0^r \sigma r^2 dr}{r^2}, \tag{1}$$

and will vanish at the boundary surface $r = r_0$ since the total charge is zero. The conditions for equilibrium reduce at once to two:

$$\frac{\partial}{\partial r}\left(-\frac{\varphi_+{}^2\sigma_+{}^2}{\pi} + \psi_+\right) + \sigma_+ F = 0; \qquad \frac{\partial}{\partial r}\left(-\frac{\varphi_-{}^2\sigma_-{}^2}{\pi} + \psi_-\right) + \sigma_- F = 0$$

when F has the the value given above. These equations show that the radial distributions of electrical densities and substance coefficients in equilibrium can be taken at pleasure for both the proton and electron, but that then the atomic potentials are uniquely determined. It is to be recalled that the arbitrary constants must be so taken that ψ_+, ψ_- vanish at the boundaries of the proton and electron, respectively.

It seems clear that the simplest possible assumption to make in the choice of the arbitrary densities is that they are constant throughout and numerically equal; in this case the electrical force F vanishes everywhere, and the conditions for equilibrium reduce to

$$\psi_+ = \frac{\varphi_+{}^2 k^2}{2\pi} + \alpha_+, \qquad \psi_- = \frac{\varphi_-{}^2 k^2}{2\pi} + \alpha_- \tag{2}$$

where k is the common numerical value of σ_+ and σ_-. On account of the fact that the atomic potentials and substance coefficients remain constant along the world line of any point-particle of the proton or electron, these functional relations (2) between ψ_+ and φ_+, and ψ_- and φ_- must hold identically. It is to be noted that φ_+ and φ_- still remain undetermined, although later a definite choice will be specified. It will be observed that thus far we have simply demanded a completely neutral stable state of the atom under a uniform charge.

2. *The Equations of Oscillation.*—Let now X, Y, Z denote the electrical forces in the directions of the x, y, z axes, respectively, while L, M, N denote the corresponding magnetic forces. If we eliminate L, M, N in the Maxwell-Lorentz equations we obtain relations of the familiar type

$$\frac{\partial^2 X}{\partial t^2} = \Delta X - 4\pi \frac{\partial \sigma}{\partial x} - 4\pi \frac{\partial u}{\partial t}, \tag{3}$$

$$\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z} = 4\pi\sigma, \tag{4}$$

where Δ denotes the Laplacian operator and u, v, w denote the current densities. These equations hold at velocities small in comparison with that of light. Since all the variables vanish in the equilibrium state, they will be small quantities for slightly disturbed motions.

Similarly, if we write down the conditions for the conservation of momentum we obtain for the proton three equations of the type

$$\frac{\bar{\rho}_+}{\bar{\sigma}_+} \frac{\partial u_+}{\partial t} = \frac{\partial}{\partial x}\left(- \frac{\bar{\rho}_+}{\bar{\sigma}_+} \sigma_+ - \frac{\bar{\rho}_+}{\bar{\varphi}_+} \varphi_+ + \psi_+ \right) + \bar{\sigma}_+ X,$$

where the bars indicate the values of the variables in the equilibrium state and where σ_+, φ_+, ψ_+ refer now to the small changes in these variables. But from the first equation (2) we find at once by the method of variation

$$\psi_+ = \frac{\bar{\varphi}_+ k^2}{\pi} \varphi_+ = \frac{\bar{\rho}_+}{\bar{\varphi}_+} \varphi_+,$$

where ψ_+, φ_+ are defined as stated. Hence the terms in ψ_+, φ_+ of the preceding equation of motion cancel, and that equation simplifies to

$$\frac{\partial u_+}{\partial t} = - \frac{\partial \sigma_+}{\partial x} - 2 \frac{\bar{\varphi}_+'}{\bar{\varphi}_+} \frac{x\sigma_+}{r} + \frac{\pi}{\bar{\varphi}_+^2} X,$$

where the accent indicates differentiation with respect to r. By combining this equation for the proton with the analogous one for the electron we obtain

$$\frac{\partial u}{\partial t} = - \frac{\partial \sigma}{\partial x} - 2 \frac{\bar{\varphi}'}{\bar{\varphi}} \frac{x\sigma}{r} + \frac{\pi}{\bar{\varphi}^2} X, \tag{5}$$

provided that we assume that φ_+ and φ_- are in a constant ratio throughout the atom in the position of equilibrium. Here $\bar{\varphi}$ is defined by the equation

$$\frac{1}{\bar{\varphi}^2} = \frac{1}{\bar{\varphi}_+^2} + \frac{1}{\bar{\varphi}_-^2}. \tag{6}$$

Thus at this stage there remains still to be determined the arbitrary function $\bar{\varphi}$ and the constant ratio $\bar{\varphi}_+/\bar{\varphi}_-$. We shall assume that the substance coefficient of the electron is comparatively *very small*, thereby requiring $\bar{\varphi}$ to be very small also.

Substituting the expression (5) in (3) we obtain

$$\frac{\partial^2 X}{\partial t^2} = \Delta X - \frac{4\pi^2}{\bar{\varphi}^2} X + 8\pi \frac{\bar{\varphi}'}{\bar{\varphi}} \frac{x\sigma}{r}, \tag{7}$$

with two similar equations in Y and Z. Here σ is defined through (4) in terms of X, Y, Z.

Finally, then, if we replace the variables X, Y, Z by Xe^{ipt}, Ye^{ipt}, Ze^{ipt}, where $p/2\pi$ is the frequency, we obtain the three frequency equations valid within the atom, such as

$$\Delta X + \left(p^2 - \frac{4\pi^2}{\bar{\varphi}^2}\right) X + 2\frac{\bar{\varphi}'}{\bar{\varphi}} \frac{x}{r}\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}\right) = 0, \tag{8}$$

while the corresponding equations in empty space are of the form

$$\Delta X + p^2 X = 0. \tag{8'}$$

There are obvious boundary conditions upon the solutions sought such as that X, Y, Z are to be finite, and continuous everywhere, and are to vanish suitably at infinity; furthermore σ is to vanish for $r \geqq r_0$ in empty space.

3. *On a Limiting Approximation.*—In order to determine the fundamental frequencies of the atom, it is necessary now to assume a particular form for $\bar{\varphi}$ which we shall define by means of the equation

$$\frac{4\pi^2}{\bar{\varphi}^2} = \alpha^2 - \frac{2\beta\sqrt{\alpha}}{r} + \frac{\gamma}{r^2} \qquad (\alpha\gamma > \beta^2), \tag{9}$$

where α is to be an exceedingly large positive constant, whose magnitude we shall not attempt to determine.

It is obvious that for α sufficiently large $\bar{\varphi}'/\bar{\varphi}$ becomes, in general, very small within the atom. Thus we are led to simplify (8) to the approximate form

$$\Delta X + \left[p^2 - \alpha^2 + \frac{2\beta\sqrt{\alpha}}{r} - \frac{\gamma}{r^2}\right] X = 0. \tag{10}$$

We propose then to search for the *everywhere finite* solutions of these equa-

tions (10), (8′). Of course, the equations (10) are of the same general type as the Schrödinger "wave equation," although the origin and significance in his theory are entirely different.

We propose, furthermore, to treat the radius of the electron as infinite. This seemingly paradoxical result may be explained as follows. Let us determine the constant β by use of Planck's constant as follows:

$$\beta^2 = \frac{8\pi^3 me^4}{h^3}$$

where m, e and h have the usual significance in C. G. S. units. This gives β^2 of the order 10^{15}. If we take units of length smaller in the ratio 1 to $\beta\sqrt{\alpha}$, (10) takes the form

$$\Delta X + \left(K + \frac{2}{r} - \frac{\gamma}{r^2}\right) X = 0. \qquad (10')$$

Hence if $1/\beta\sqrt{\alpha}$, light-seconds is small in comparison with the atomic radius, the atomic radius will be large for the units employed in (10′).

It will be assumed that the absolute constant γ is not large.

4. *Determination of Frequencies.*—The everywhere finite solutions of (10) may be found as follows. Since a typical form of disturbance of the atom is that introduced by the removal of a uniform field of force in the direction of the z axis for instance, it is natural to write

$$X = \frac{x}{r}F, \qquad Y = \frac{y}{r}F,$$

where the radial electric force F satisfies the equation

$$r\Delta\left(\frac{1}{r}F\right) + 2\frac{\partial}{\partial r}\left(\frac{1}{r}F\right) + \left(p^2 - \alpha^2 + \frac{2\beta\sqrt{\alpha}}{r} - \frac{\gamma}{r^2}\right)F = 0,$$

and depends only on the variables r, φ (r, φ, ψ being spherical coördinates). A solution $F = R(r)\Phi(\varphi)$ is single valued in φ only if Φ is a Legendre's polynomial of the nth degree in which case R satisfies the equation,

$$R'' + \frac{2}{r}R' + \left(p^2 - \alpha^2 + \frac{2\beta\sqrt{\alpha}}{r} - \frac{n(n+1) + \gamma + 2}{r^2}\right)R = 0.$$

Essentially only one solution, R, will be finite at $r = 0$ for any value of p and α, namely, that solution which belongs to the positive exponent μ given by the characteristic equation

$$\mu(\mu + 1) = n(n + 1) + \gamma + 2. \qquad (11)$$

If it is also to vanish suitably at $r = +\infty$ it must have the form

$$r^\mu e^{-qr} P(r)$$

where P is a polynomial of degree l ($l = 0, 1, 2, \ldots$) satisfying the equation

$$rP'' + 2(\mu + 1 - qr)P' + 2(b\sqrt{\alpha} - (\mu + 1)q)P = 0$$

of the Laplace type and where $q^2 = \alpha^2 - p^2$. The lth derivative P_l satisfies a like equation (obtained by l differentiations) in which the coefficient of P_l must vanish. This gives the approximate formula

$$p = \alpha - \frac{\beta^2}{2(l + 1/2 + \sqrt{(n + 1/2)^2 + \gamma + 2})^2} \quad (l, n = 0, 1, 2, \ldots)$$

where γ is small. Hence we see that a high limiting frequency is present as suggested by Schrödinger.[3] The only finite corresponding Z is 0.

On the other hand we may take $X = Y = 0$ and then determine Z to be finite, in which case the frequencies are readily found to be the same as before except that $\gamma + 2$ is replaced by γ. It is this type of oscillation which has physical importance, since the nearby passage of an electron would produce it. In this case if γ is neglected the difference frequencies of the Balmer formula appear with

$$\nu = \frac{2\pi^2 me^4}{h^3}\left(\frac{1}{(1 + l)^2} - \frac{1}{(1 + l')^2}\right)$$

where l, l' ($l > l'$) are any pair of distinct integers.

5. *Concluding Remarks.*—Consider a proton and coincident electron at rest according to the theory developed above. If these are disturbed, the electron responds at first as though it were free, precisely because of the complete initial neutralization of the fields due to the charges of the proton and electron. This is especially true near the center of the electron. But elsewhere the electron responds more slowly, of course, and thus a vibratory motion of stationary type is set up within the electron. This is characterized by the nodal numbers l and n of the above theory.

The process of dissipation of the energy of such a stationary wave is probably somewhat complicated.

It is desirable to remark that the precise form of the substance coefficients selected above appears to be only an incidental feature. The essential requirements appear to be two: (1) complete neutralization in equilibrium, i.e., $\sigma_+ + \sigma_- = 0$; (2) a combined substance coefficient $\bar{\varphi}$ in general very small and vanishing at the center of the atom.

[1] The particular specialization made was suggested by the form of the "wave equation" used by Schrödinger in his papers "Quantisierung als Eigenwertproblem," *Annalen der Physik*, vol. 79, 1926.

[2] In general the notation is that of the preceding note unless otherwise stated. The subscripts + and − refer to the proton and electron, respectively, in all cases.

[3] Compare Schrödinger, loc. cit. 1.

GROUPS GENERATED BY TWO OPERATORS OF ORDER THREE WHOSE PRODUCT IS OF ORDER SIX

BY G. A. MILLER

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS

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Let s_1, s_2 represent two operators which satisfy the following conditions: $s_1^3 = s_2^3 = (s_1s_2)^6 = 1$. If at least one of the three operators s_1, s_2, s_1s_2 is the identity the order of the group, G, generated by s_1 and s_2 cannot exceed 3. Hence we shall assume in what follows that each of the two operators s_1, s_2 is of order 3 and that s_1s_2 is not the identity. If s_1s_2 is of order 2 it is well known that G must be the tetrahedral group. The case when s_1s_2 is of order 3 was considered briefly by W. E. Edington.[1] Later it was considered by the present writer from a different point of view.[2] It seems desirable to note here several fundamental results which were not explicitly stated in the articles to which reference has just been made. The three operators $s_1s_2^2$, $s_2^2s_1$, $s_1^2s_2^2s_1^2$, which constitute a complete set of conjugates under G, must be distinct unless s_1 and s_2 are commutative. A necessary and sufficient condition that G is non-abelian is, therefore, that g, the order of G, exceeds 9. When this condition is satisfied and the order of $s_1s_2^2$ is not divisible by 3 the central of G must be the identity.

The order of the largest group which s_1, s_2 can generate when s_1s_2 is of order 3 is known to be three times the square of the order of $s_1s_2^2$, and some of the fundamental properties of this group have been determined. It is often useful to know the other possible groups which can be generated by two such operators. When the order of $s_1s_2^2$ is prime to 3 and none of the prime factors of this order is of the form $1 + 3k$, where k is an integer, then it is obvious that the two cyclic groups generated by $s_1s_2^2$ and $s_2^2s_1$, respectively, can have only the identity in common, and hence the group generated by s_1, s_2 is completely determined by the order of $s_1s_2^2$. In all other cases the possible groups are also completely determined by their orders but the order of such a group is not determined by the order of the operator $s_1s_2^2$. In particular, when 3^α is the highest power of 3 which divides the order of this operator, $\alpha > 0$, then the index of the highest power of 3 which divides g is either 2α or $2\alpha + 1$, and both of these values are possible. On the other hand, if the order of $s_1s_2^2$ is divisible by p^α, where p is a prime number of the form $1 + 3k$, then the index of the highest power of p which divides g may vary from α to 2α, and each of these values corresponds to a possible group. *These results represent every possible group that can be generated by two operators of order 3 whose product is also of this order.* It may be added that the same infinite system of non-abelian groups is defined by the fact that it is composed of the groups which can

be generated by two operators of order 3 whose product has exactly three conjugates under the group generated by these two operators.

It remains to consider the case when s_1s_2 is actually of order 6, and in what follows it will be assumed that this condition is always satisfied. This operator of order 6 generates an operator s of order 3 and an operator t of order 2. One would expect that the general problem would be greatly simplified by assuming successively that one or the other of these two operators is invariant under G. If it is assumed that s is invariant under G, while no condition is imposed on t, the considerations become very simple, and hence we shall begin with this case. That is, we shall assume for the present that the operators s_1, s_2 satisfy also the condition

$$(s_1s_2)^2 = (s_2s_1)^2 = s.$$

When this condition is satisfied the two operators s_1s_2 and s_2s_1 are commutative. In fact, $s_2s_1.s_1s_2 = s^2s_1^2s_2^2.s_2^2s_1^2 = s^3s_1s_2^2s_1 = s_1s_2^2s_1$. Since $s_1^2s_2s_1^2$ is the inverse of $s_1s_2.s_2s_1$ it results that the group of order 4 generated by $(s_1s_2)^3$, $(s_2s_1)^3$ is invariant under G but its operators of order 2 are not commutative with s_1. Hence this four-group and s_1 generate the tetrahedral group, and we have established the following theorem: *If two operators of order three have a product of order six whose square is invariant under at least one of these two operators then the group generated by these operators is the direct product of the group of order three and the tetrahedral group.* It may be explicitly noted that two such operators can generate only one group and hence it results that any additional condition imposed on these operators must be redundant or lead to some contradiction. This group can also be generated by two operators of order 3 whose product is of order 3.

If s_1, s_2 satisfy the following four conditions:

$$s_1^3 = s_2^3 = (s_1s_2)^6 = 1, \qquad (s_1s_2)^3 = (s_2s_1)^3 = t$$

and s_1s_2 is actually of order 6 the considerations become much more difficult than in the case just considered, but they remain sufficiently simple for an exhaustive treatment. It will be convenient to employ two additional symbols with the following definitions:

$$s_1s_2^2 = t_1, \qquad s_2^2s_1 = t_2.$$

The three conjugates of t_1 under powers of s_1 are, therefore, t_1, t_2, $t^{-1}_2t^{-1}_1$. The conjugates of the same operator under powers of s_2 are t_1, t_2, $t\,t^{-1}_2t^{-1}_1 = t^{-1}_1t^{-1}_2$. Hence t_1 has always exactly six distinct conjugates under the group, G, generated by s_1, s_2. That is, if any additional conditions are imposed on s_1, s_2, which are consistent with those noted above, they cannot

affect the number of these distinct conjugates. In particular, t_1 cannot be the identity.

As t_1 and t_2 are conjugate they must have the same order, and this order cannot be odd since one transforms the other into itself multiplied by t. We proceed to prove that this order must be divisible by 4. From the fact that $t_1^2 t_2 = t_2 t_1^2$ and $t_1 t_2^2 = t_2^2 t_1$ it results that all the operators of odd order generated by t_1 and t_2 are invariant under the group generated by t_1 and t_2. Hence the largest operators whose orders are a power of 2 which are generated by t_1 and t_2, respectively, cannot be commutative. If these operators were of order 2 they would, therefore, generate the octic group, and this group would be invariant under G. This is impossible since the octic group does not admit an automorphism of order 3 and all its operators cannot be commutative with s_1 and s_2. It is, therefore, necessary that the common order of t_1 and t_2 is divisible by 4, and it will be proved that s_1 and s_2 can be so selected that this common order is an arbitrary multiple of 4.

With a view toward a proof of this statement we shall first consider the group H of largest possible order which can be generated by two operators satisfying the conditions which have been imposed on t_1 and t_2. This order is obviously twice the square of the common order of t_1 and t_2, and H is the direct product of its Sylow subgroups. All of these subgroups are abelian except the one of order 2^m, and each of them involves a set of two independent generators of a common order. Evidently m is odd and the index of every other power of a prime which represents the order of such a Sylow subgroup is even. The group of inner isomorphisms of H is the four-group. The central of H is composed of all its operators whose orders are not divisible by $2^{\frac{m-1}{2}}$. Hence H admits an automorphism in which

$$t_1 \sim t_2,\ t_2 \sim t_2^{-1} t_1^{-1},\quad t_2^{-1} t_1^{-1} \sim t_1.$$

As this automorphism is of order 3 it may be supposed to be effected by s_1.

It is easy to see that all of the operators of G which are not also in H are either of order 3 or of order 6, and in the latter case their common cube is t. In fact, all the operators of H are of the form $t_1^\alpha t_2^\beta t^\gamma$ and

$$(s_1 t_1^\alpha t_2^\beta t^\gamma)^3 = t^\gamma.$$

Moreover, just half of these operators are of order 3 while the other half are of order 6 since the products obtained by multiplying t into all of these operators must be distinct and the operators of order 3 are transformed thereby into those of order 6, and vice versa. Any of these operators, together with the central of H, generates a subgroup of index 4 under G in which the product of two operators of order 3 which appear in the same

co-set with respect to this central is of order 3, since the product of the square of one of these operators into the other is commutative with all of its conjugates under G. Each of the operators of G which is not found in H appears, therefore, once, and only once, in the four subgroups of index 4 under G which have the central of H in common but involve no other operator of H. If an operator of order 3 which does not appear in H is multiplied into another operator of order 3 which is found in the same co-set of G with respect to H the product will be of order 3 if, and only if, these two operators appear in the same subgroup of index 4 noted above, otherwise it will be of order 6, since in the latter case the product of the square of one of these operators into the other cannot be commutative with its conjugates under these operators. If these two operators of order 3 are so selected that the product of one into the square of the other is an operator of largest order in H their product must, therefore, be of order 6 and they can be used for s_1 and s_2, respectively.

Hence it results that two operators of order 3 which satisfy the conditions that their product is of order 6 and that the cube of this product is invariant under one of these operators generate a group whose order cannot exceed 6 times the square of the order of the product of one of these operators into the square of the other. Moreover, such a group of maximal order always exists whenever the order of the latter product is divisible by 4. The smallest order of such a maximal group is 96 and this group is generated by the following substitutions:

$$s_1 = aiq.bjr.clt.dks.emu.fnx.gov.hpw$$
$$s_2 = aiu.bjx.clv.dkw.ent.fms.gpq.hor.$$

The group thus generated is transitive and of degree 24. It involves an invariant subgroup of order 32 which has three transitive constituents of degree 8 and of order 16. The invariant substitution of order 2 is

$$ab.cd.ef.gh.ij.kl.mn.op.qr.st.ux.vw.$$

Some of the preceding results may be stated in the form of a theorem as follows: *If two operators of order three have a product of order six and if the cube of this product is invariant under at least one of these operators but no further restrictions are imposed on these operators then they generate a group whose order is six times the square of the order of the product of one of these operators into the square of the other.*

In what precedes we considered the largest group which can be generated by the two operators s_1, s_2 when they satisfy the given conditions. It is interesting to inquire under what conditions additional restrictions may be imposed on these operators so as to give rise to a group of lower order. When t_1 is of order 4 it is well known that s_1, s_2 can be so selected as to satisfy

the given conditions and generate the group of order 24 which does not contain a subgroup of order 12. That is, when t_1 is of order 4 the group generated by t_1, t_2 may be either the quaternion group or the group of order 32 noted, but not fully described, in the preceding paragraph. Hence in this case s_1, s_2 can be so chosen that they satisfy the given conditions and generate either the group of order 24 which involves no subgroup of order 12 or the group of order 96 whose generators are given in the preceding paragraph. It results directly from the given conditions that no other group is possible in this case.

Suppose that the highest power of 2 which divides the order of t_1 is 2^{m_1}, $m_1 > 2$, and that t_1 generates t. Since s_1 is commutative with two, and only two, operators of the Sylow subgroup of order 2^m contained in H the groups generated by t_1 and t_2, respectively, have two and only two common operators. Hence the said Sylow subgroup of order 2^m involves three, and only three, operators of order 2. This is impossible since these three operators include t and hence all of them would be commutative with s_1. The value of m must, therefore, be $2m_1 + 1$, and the Sylow subgroup of order 2^m must be completely determined by the value of m_1. It, therefore, results that this Sylow subgroup is completely determined by the order of t_1 except when this order is divisible by 4 but not by 8. The considerations relating to the odd divisors of the order of t_1 are exactly the same as in the case when s_1s_2 is of order 3. Hence the following theorem: *If two operators s_1, s_2 of order 3 have a product of order 6 whose cube is invariant under at least one of these operators and if the order of $s_1^2s_2$ is divisible by 8 but not by 3 and involves no prime factor of the form $1 + 3k$, where k is an integer, then the order of the group generated by s_1, s_2 is six times the square of the order of $s_1s_2^2$, and this group is completely determined by this order.* For all other values of the order of $s_1s_2^2$ which are multiples of 4 additional conditions may be imposed on s_1, s_2 so as to affect the group which they generate. In particular, if a prime of the form $1 + 3k$ divides this order then the Sylow subgroup of G whose order is a power of this prime may be the Sylow subgroup of this prime power order generated by $s_1s_2^2$, or it may have a second independent generator whose order is an arbitrary divisor of this prime power order. All of these groups, and no others, can be generated by s_1 and s_2.

[1] *Trans. Amer. Math. Soc.*, **25** (1923) 208.

[2] *Proc. Nat. Acad. Sci.*, **13** (1927) 24.